Matrix Lie Theory for the Roboticist

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Outline

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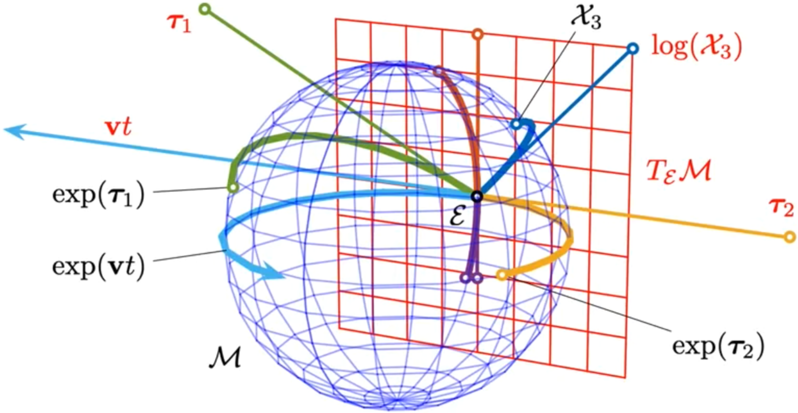
1. Presentation: Some examples
2. Overview of Lie theory
   - Lie group definition: Group, manifold, and action
   - The tangent space: Lie algebra and Cartesian
3. Operators in the Lie theory
   - The exponential and logarithmic map
   - Plus and minus operators
   - The adjoint matrix
4. Calculus and probability on Lie Groups
   - Calculus and Jacobians
   - Differentiation rules on Lie groups
   - Perturbations on Lie groups and covariance matrices
   - Integration on Lie groups
5. Applications: Localization
6. Conclusions and problems

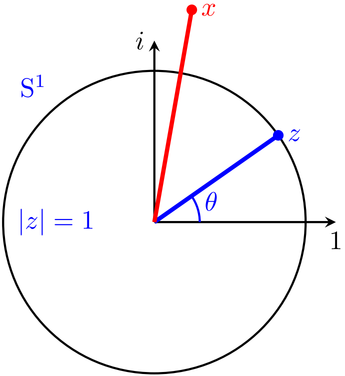
Some examples



Courtesy by Solà, J., Dery, J., and Atchuthan, D. (2021). A micro Lie theroy for state estimation in robotics.

S^1: The unit complex numbers

A quick overview of know facts



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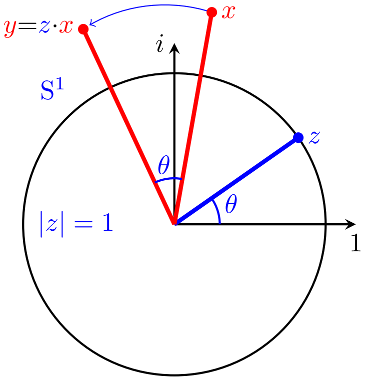

complex numbers

- Action: $y = z \cdot x$ rotates x

operator: Lie group!

- Constraint: $z^* \cdot z = 1$
- Topology: unit circle S^1
- Elements: $z = \cos\theta + i \sin\theta$
- Inverse: z^*
- Composition: $z_1 \cdot z_2$

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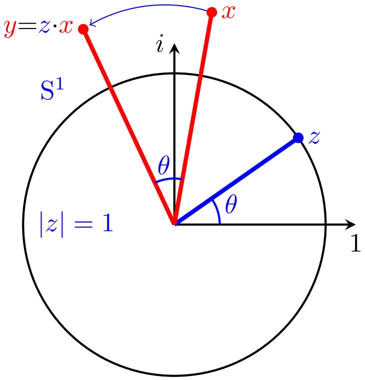

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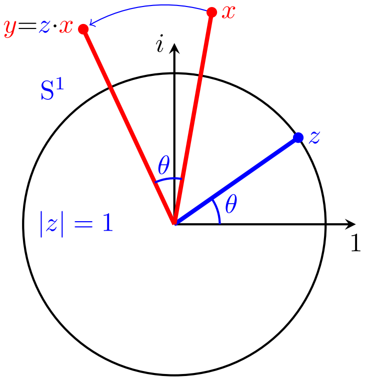

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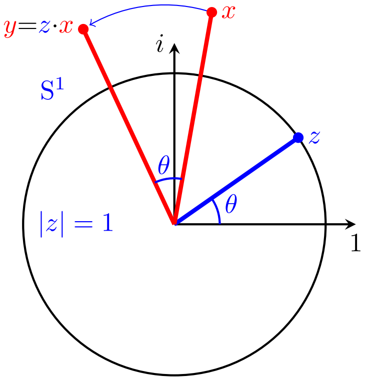

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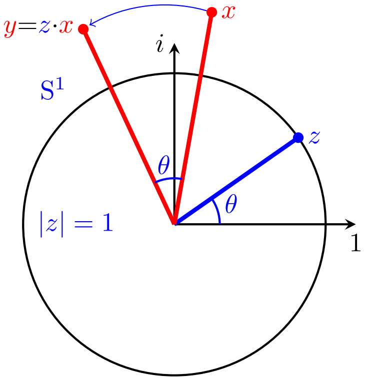

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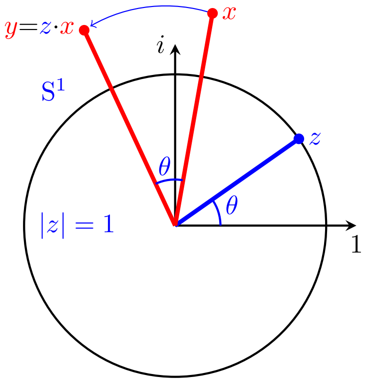

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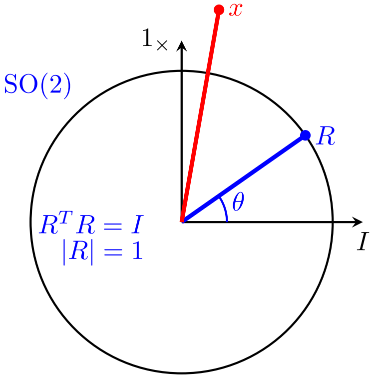

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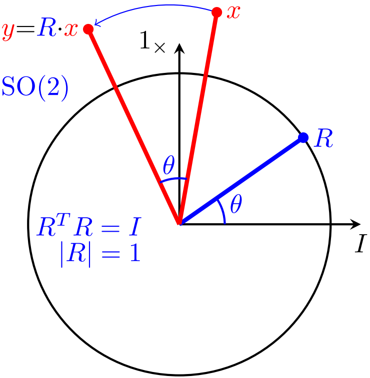

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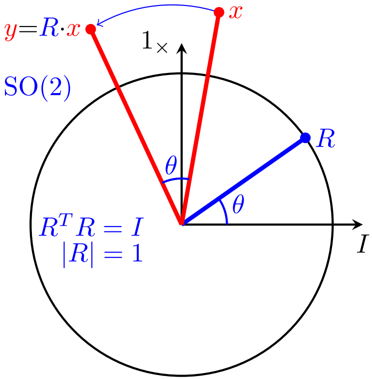

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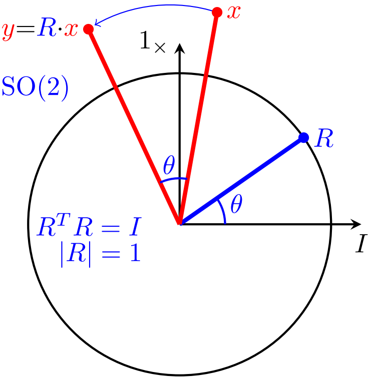

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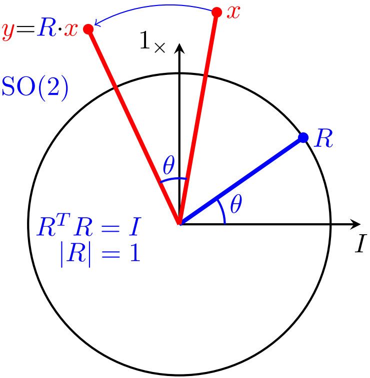

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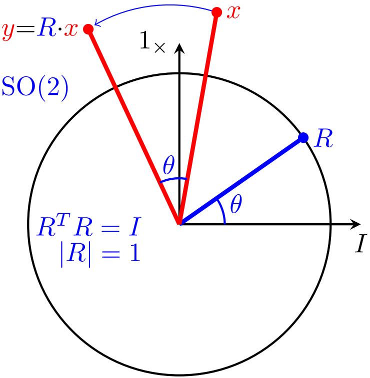

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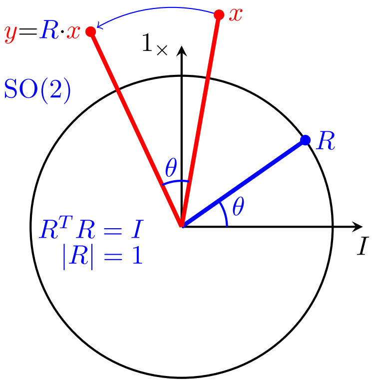

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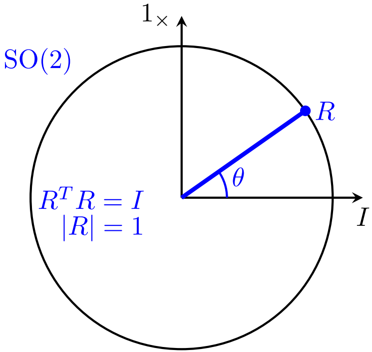
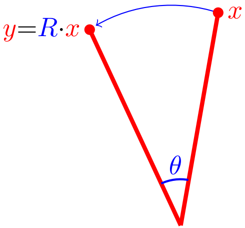


vectors: other set!

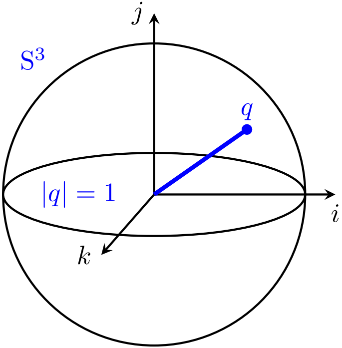


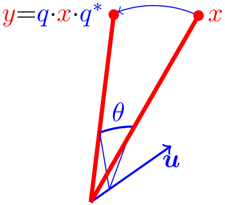
S^3: The unit quaternions

The 3-sphere in $\mathbb{R}^4$

operator: Lie group!

vectors: other set!





$$q = \cos(\theta/2) + \boldsymbol{u} \sin(\theta/2)$$

$$\boldsymbol{u} = iu_x + ju_y + ku_z$$

Typical uses

Pose of a robot in the plane: SE(2)

$$\mathcal{X}(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$

$\mathcal{X}$ is the transformation from body frame to world frame

$$R = R_{wb}$$

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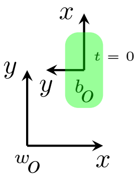

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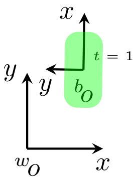

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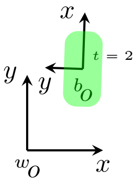

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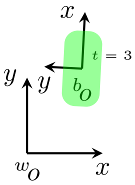

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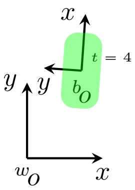

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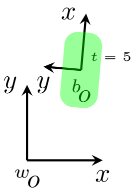

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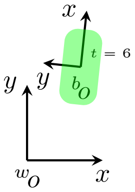

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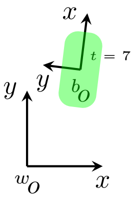

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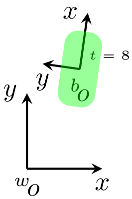

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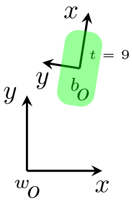

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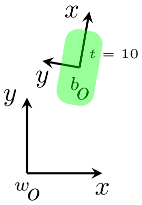

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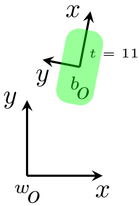

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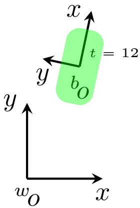

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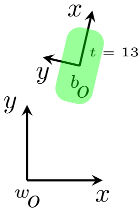

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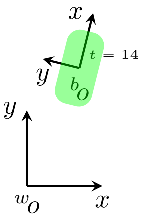

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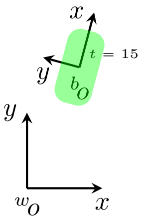

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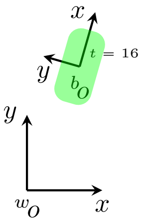

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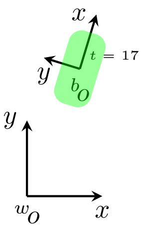

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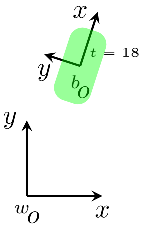

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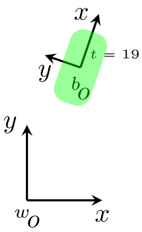

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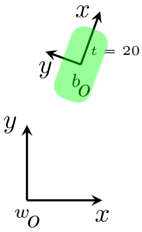

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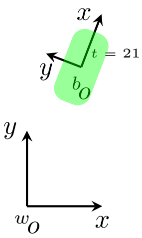

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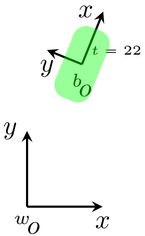

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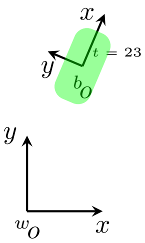

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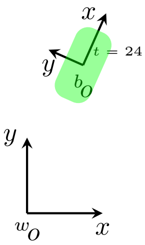

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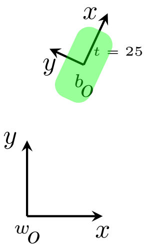

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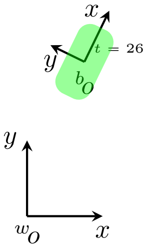

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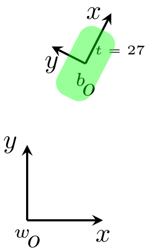

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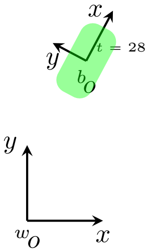

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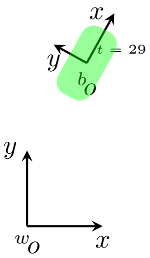

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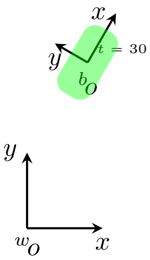

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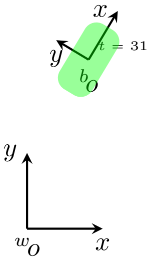

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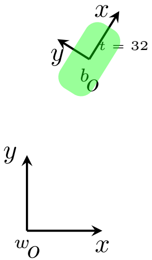

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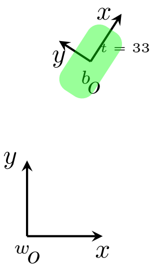

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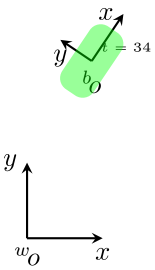

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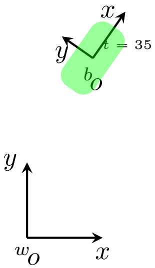

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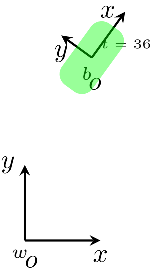

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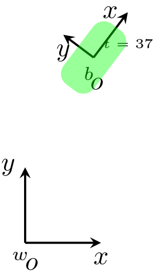

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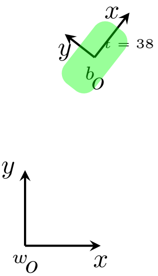

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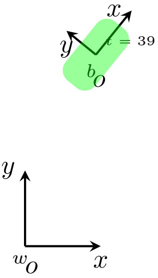

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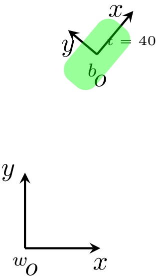

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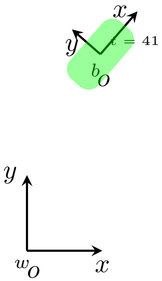

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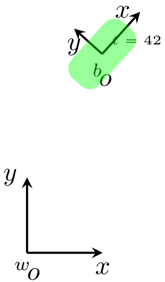

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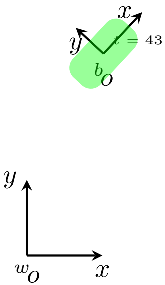

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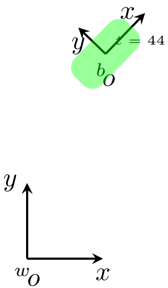

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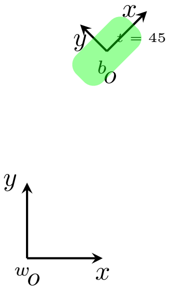

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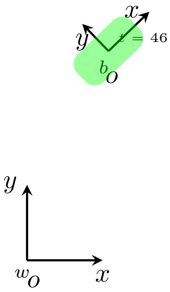

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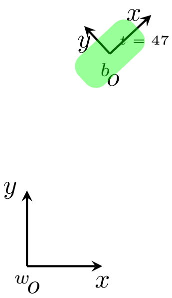

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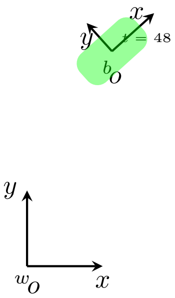

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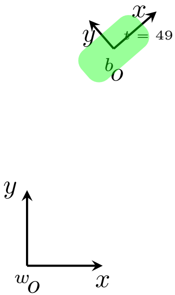

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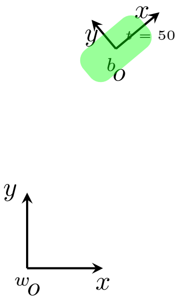

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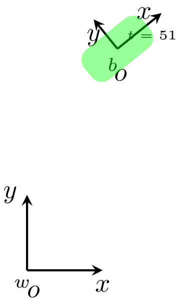

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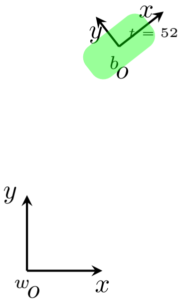

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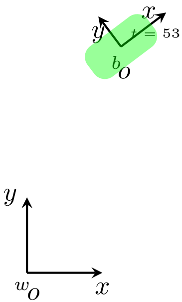

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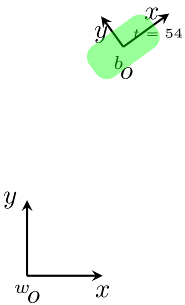

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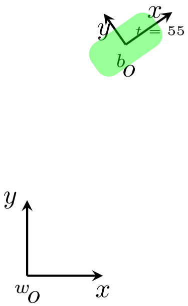

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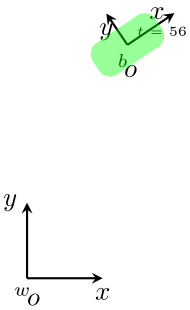

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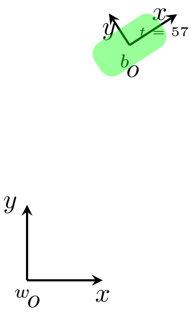

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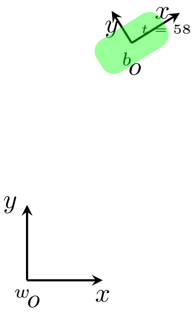

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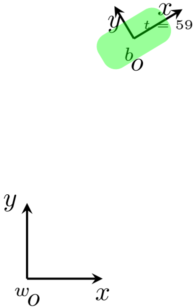

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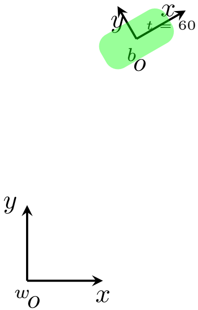

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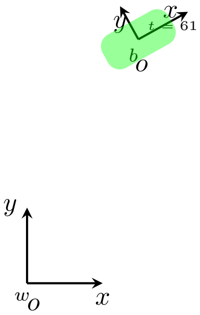

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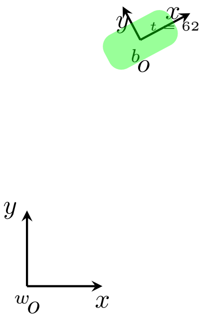

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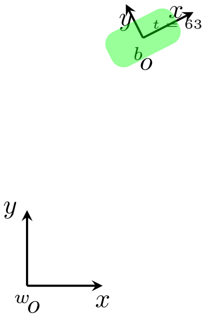

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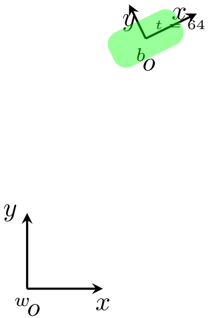

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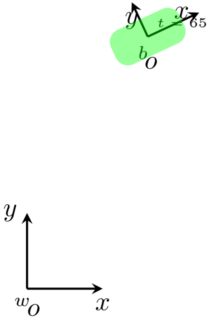

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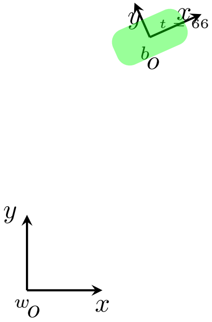

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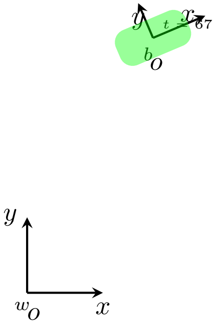

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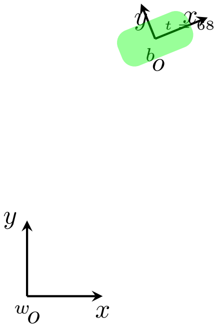

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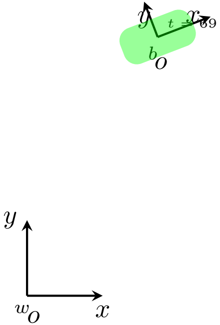

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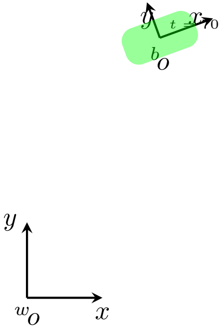

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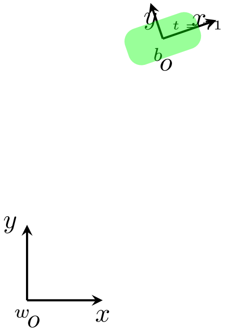

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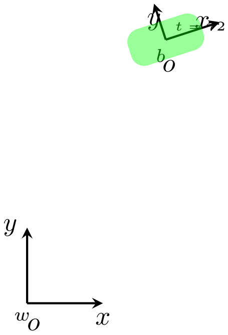

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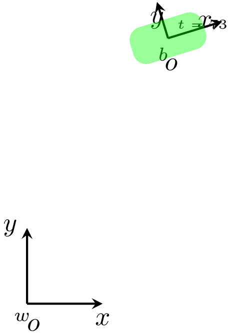

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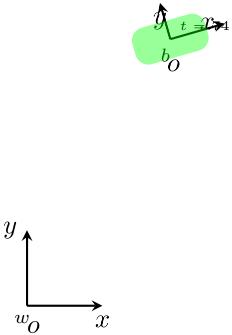

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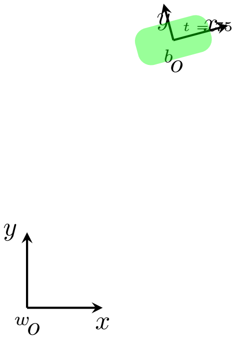

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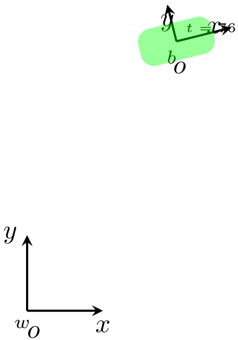

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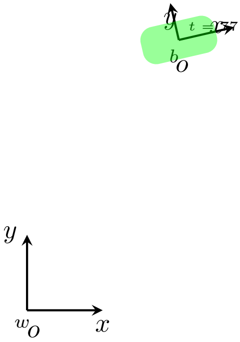

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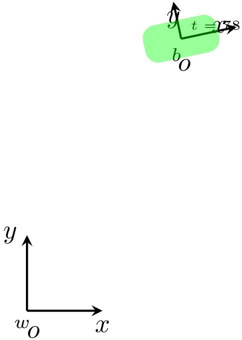

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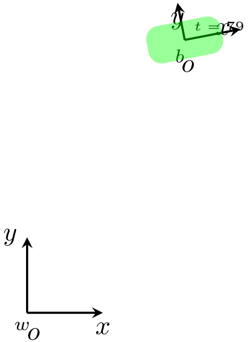

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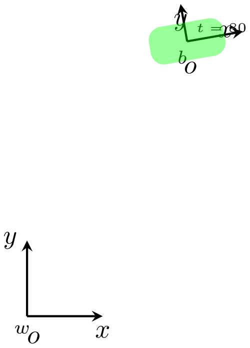

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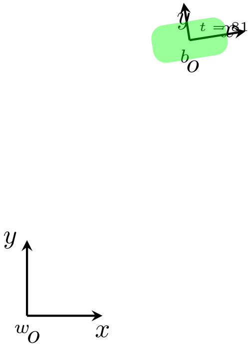

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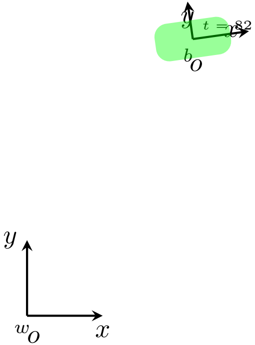

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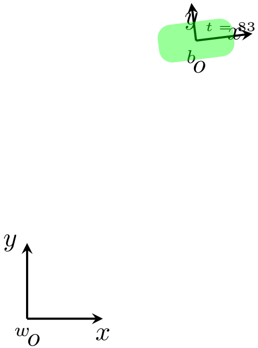

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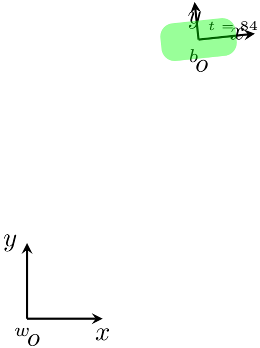

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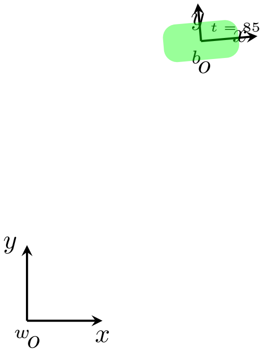

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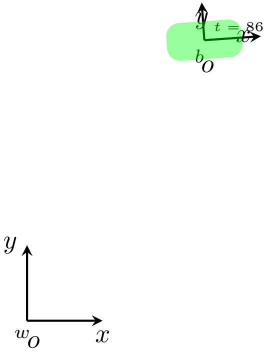

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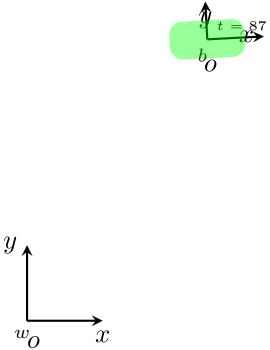

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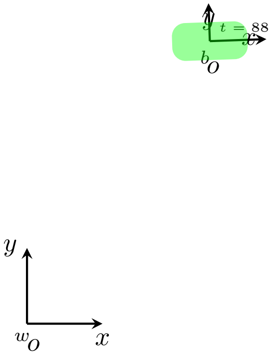

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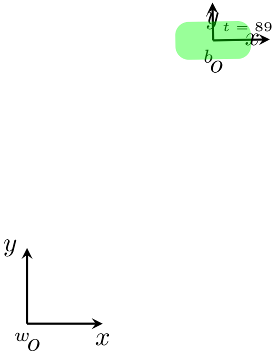

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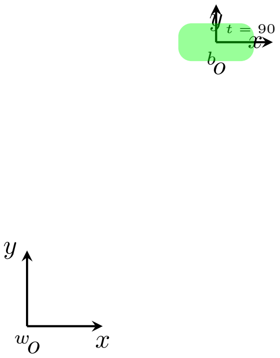

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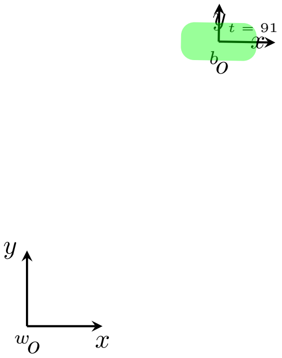

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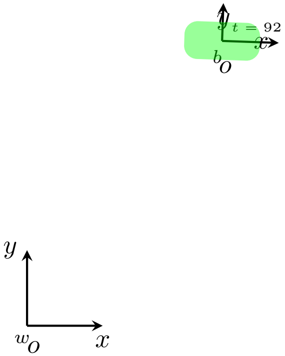

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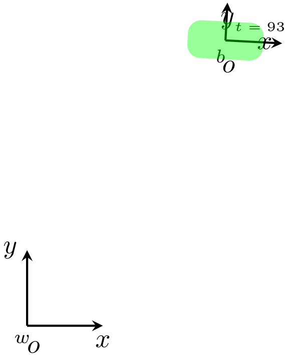

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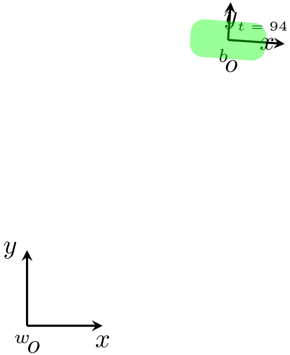

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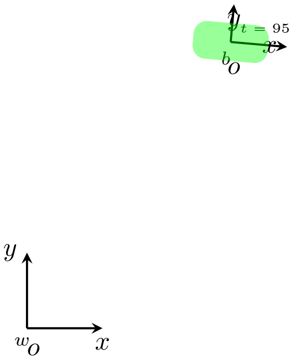

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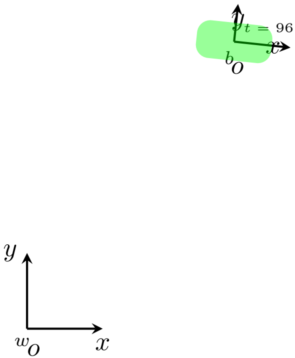

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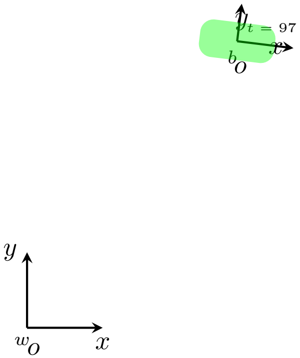

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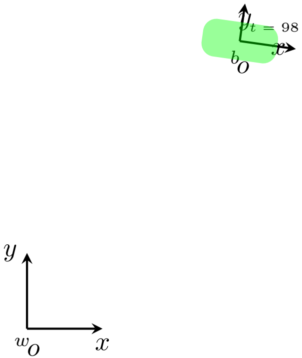

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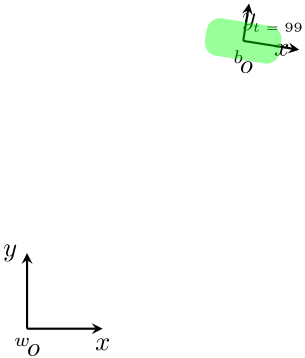

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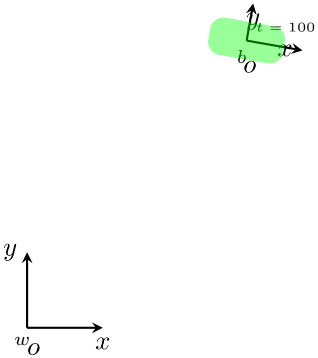

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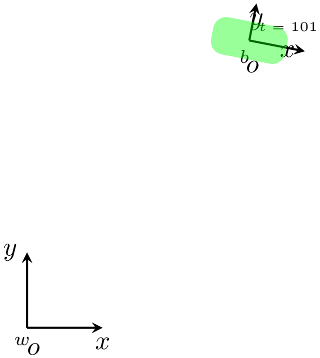

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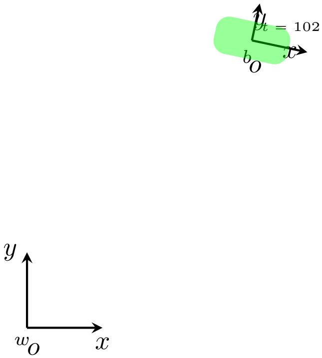

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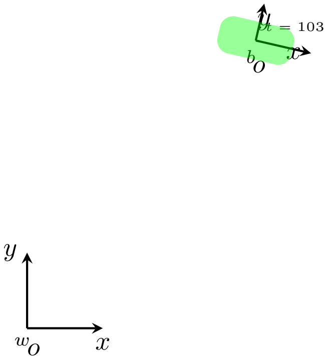

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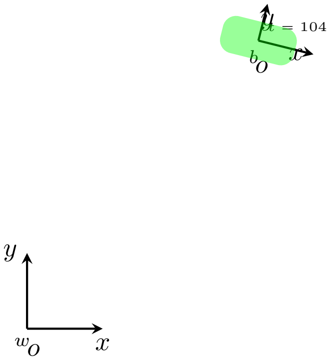

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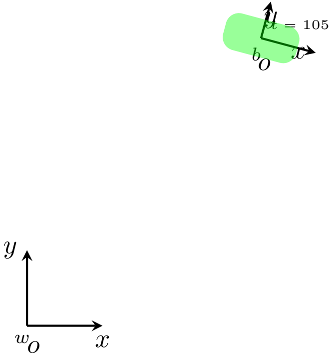

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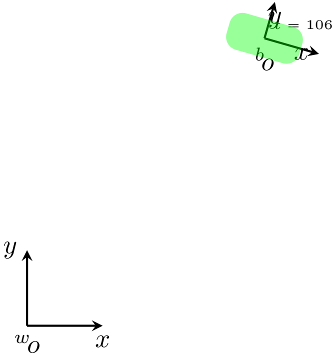

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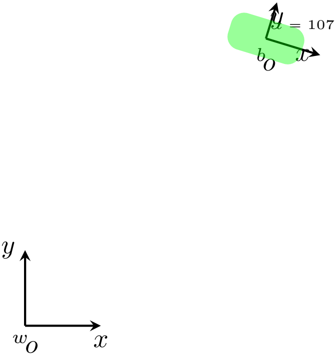

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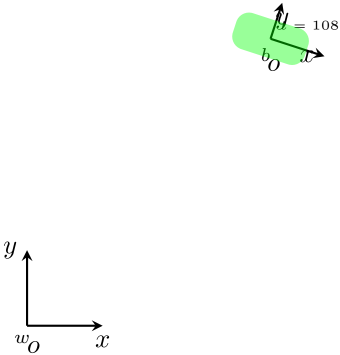

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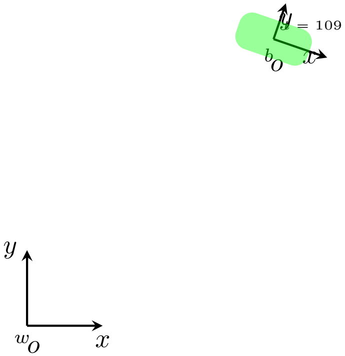

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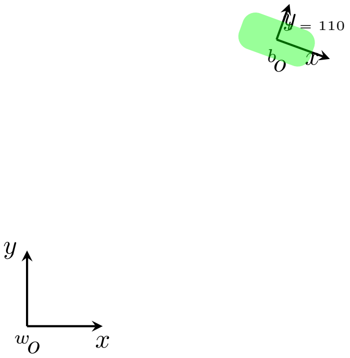

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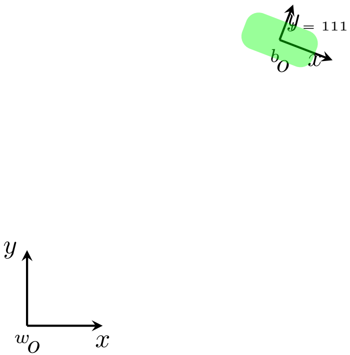

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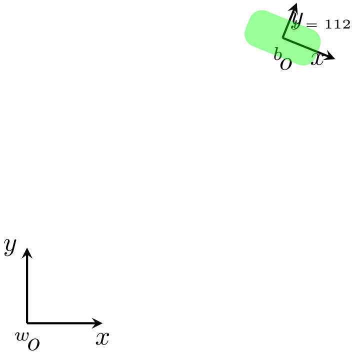

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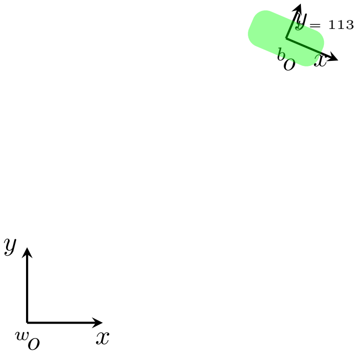

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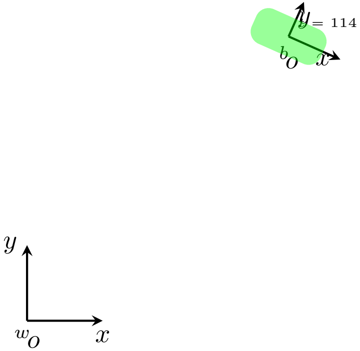

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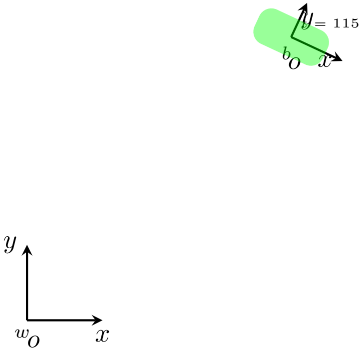

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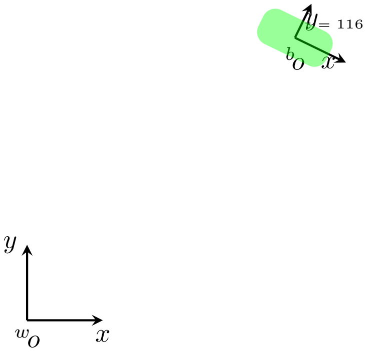

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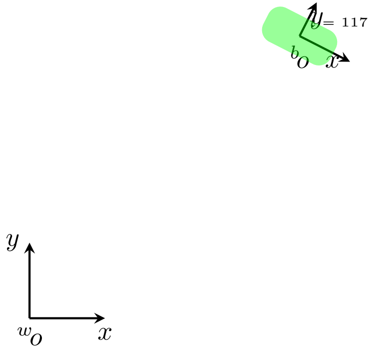

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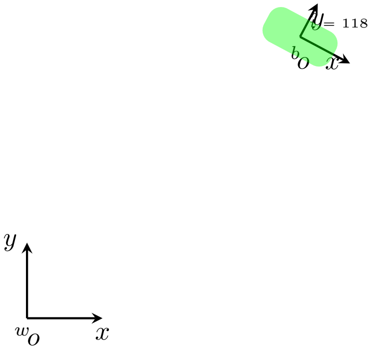

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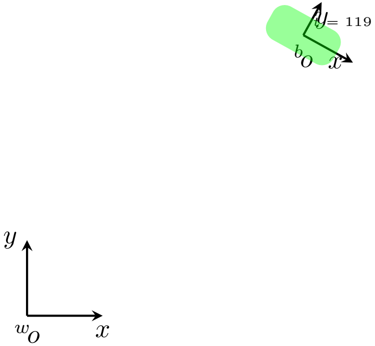

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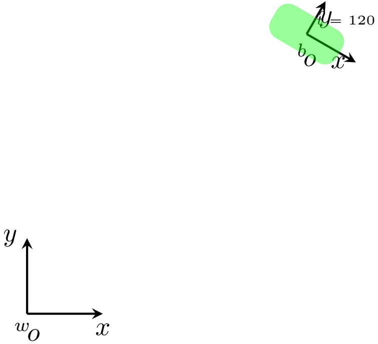

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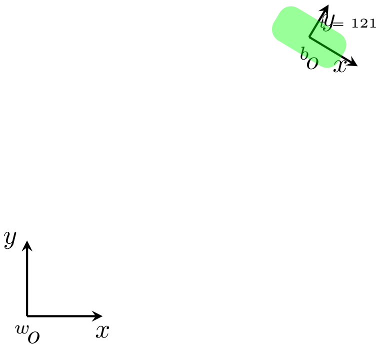

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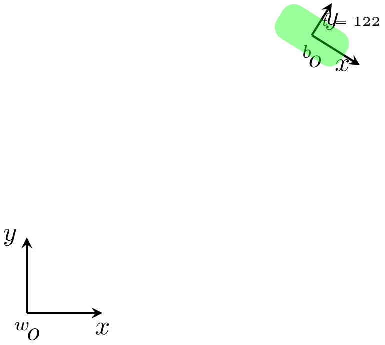

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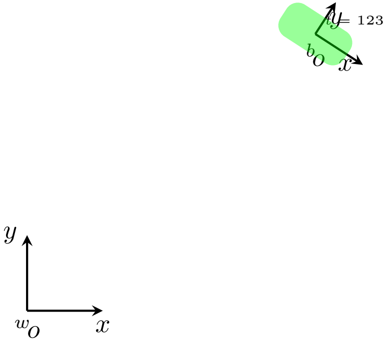

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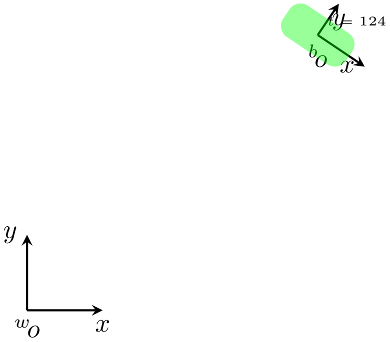

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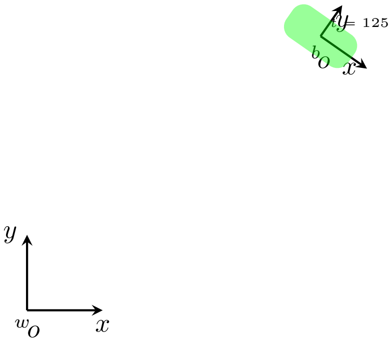

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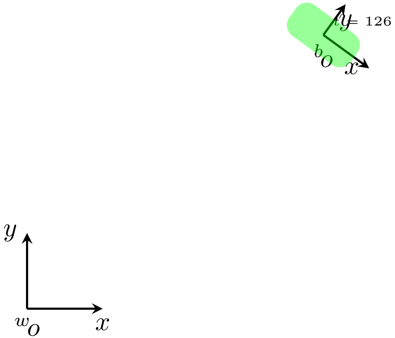

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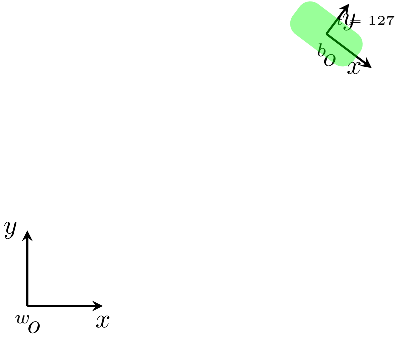

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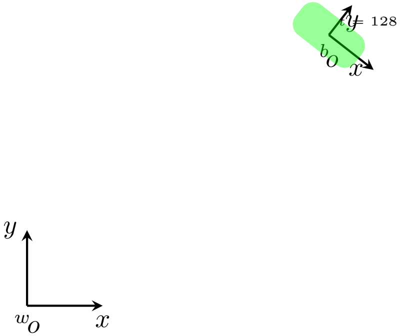

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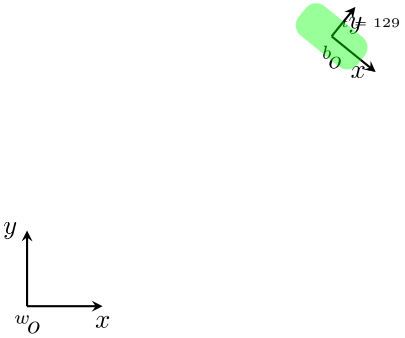

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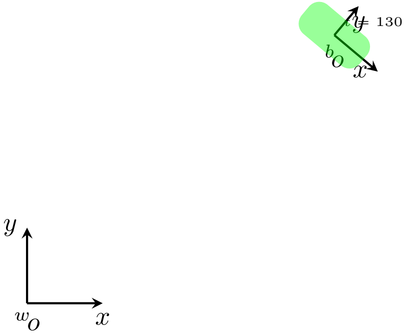

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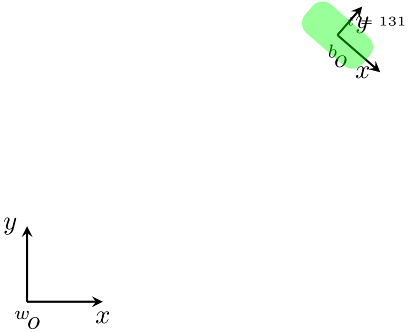

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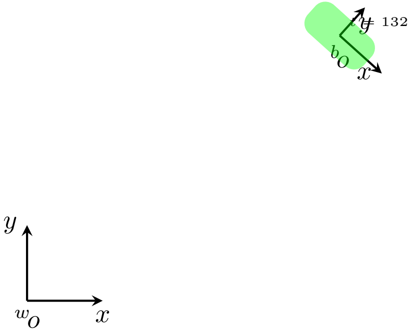

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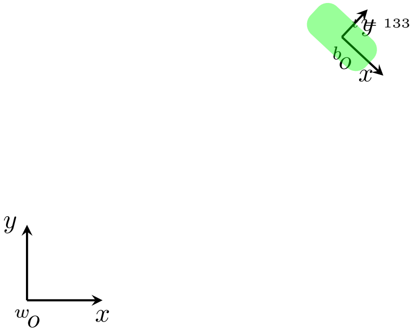

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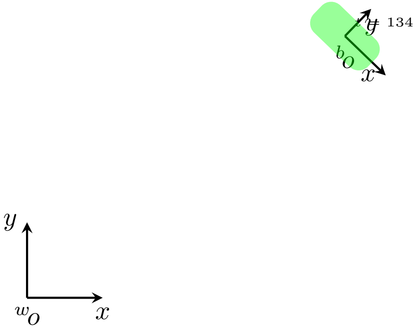

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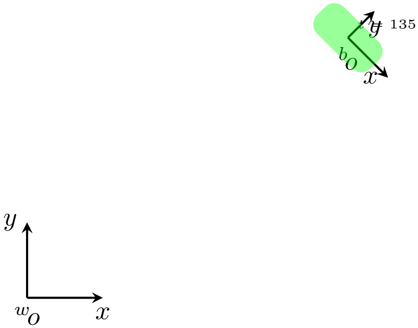

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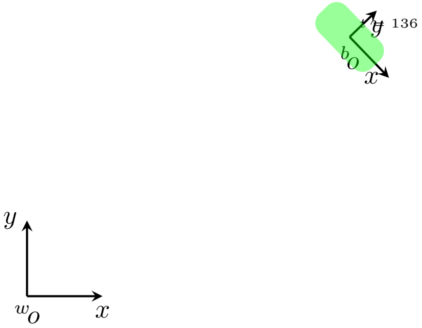

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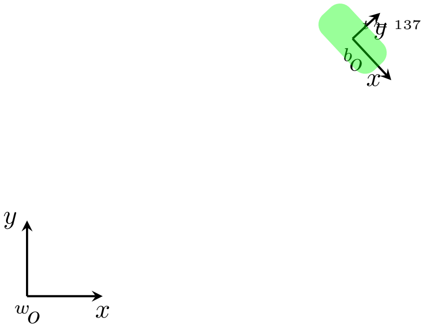

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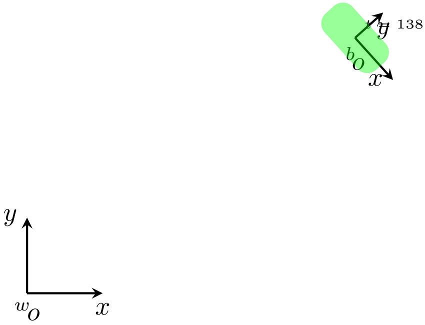

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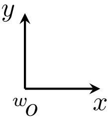

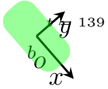

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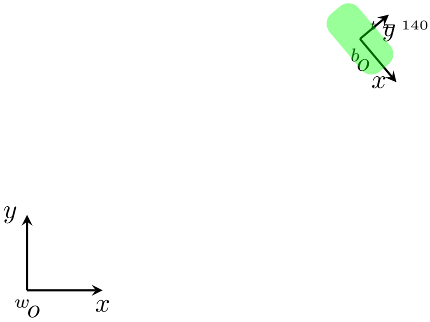

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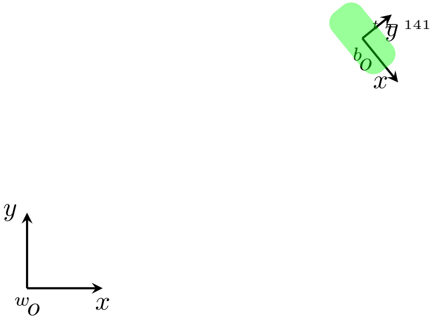

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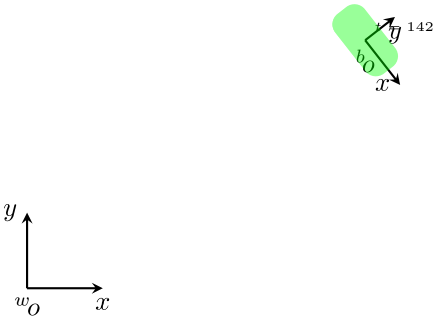

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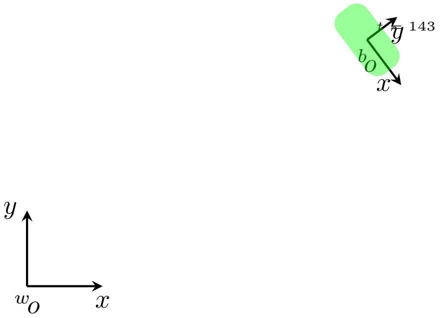

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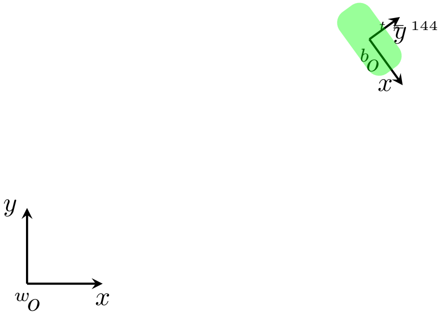

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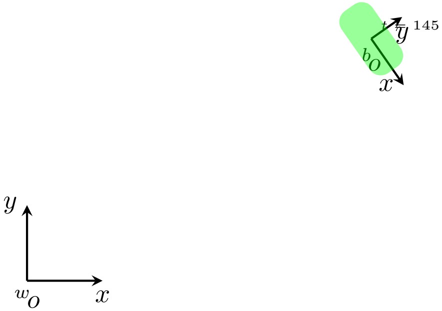

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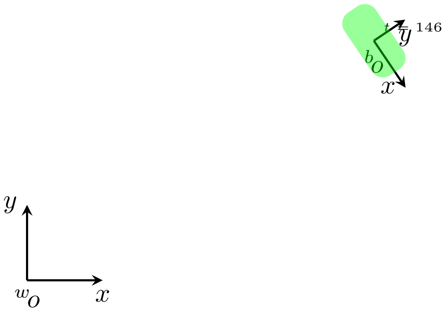

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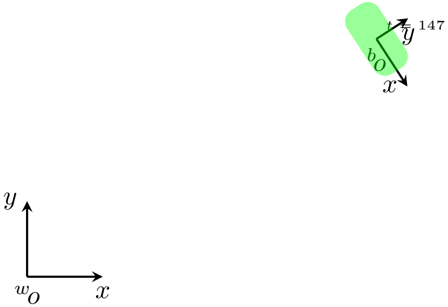

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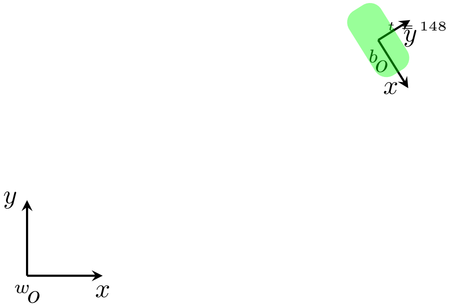

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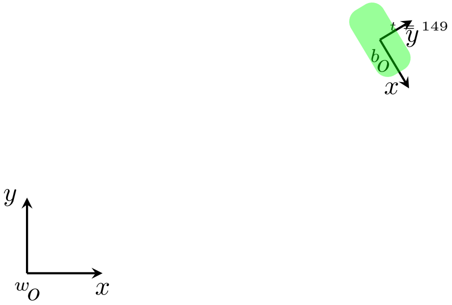

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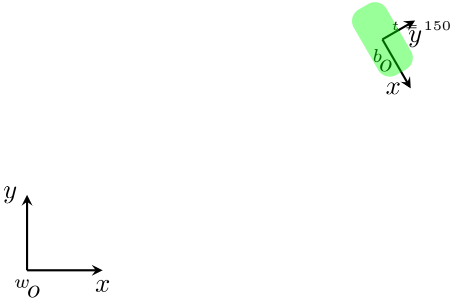

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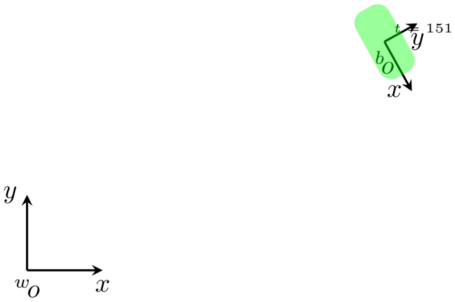

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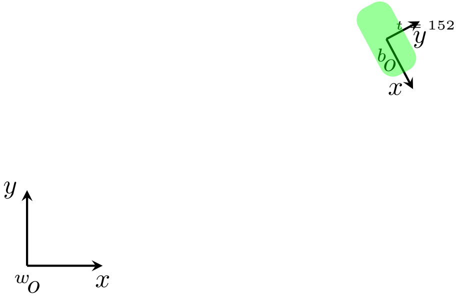

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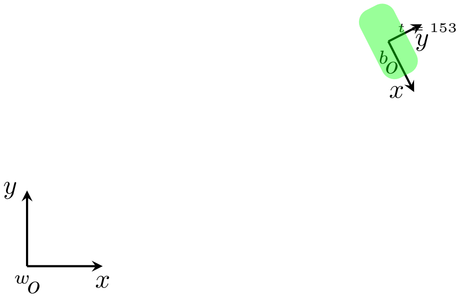

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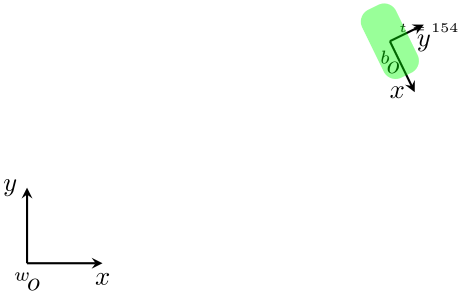

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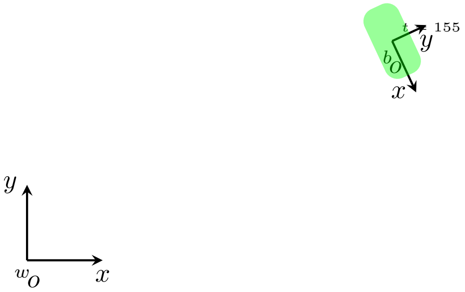

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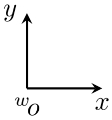

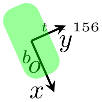

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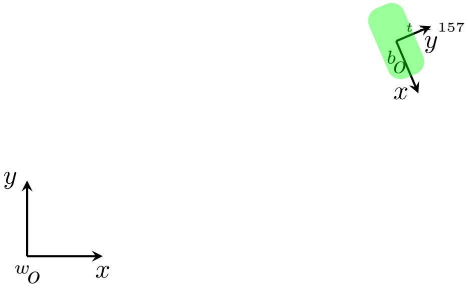

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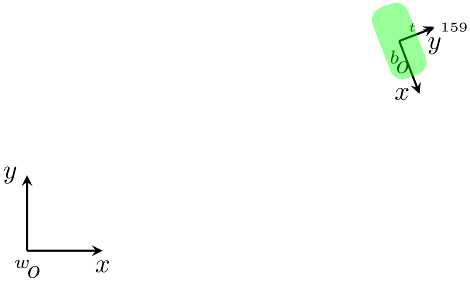

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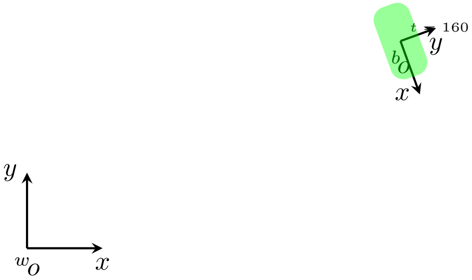

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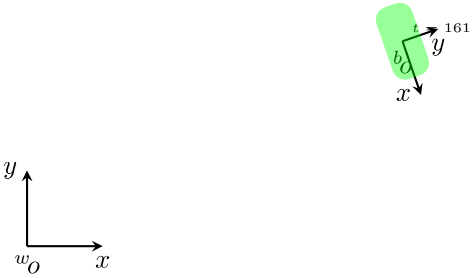

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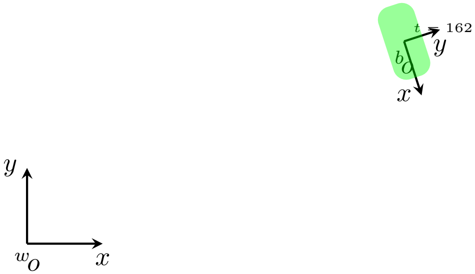

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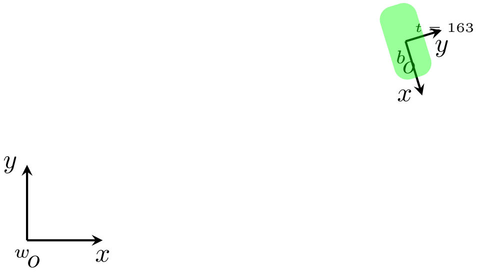

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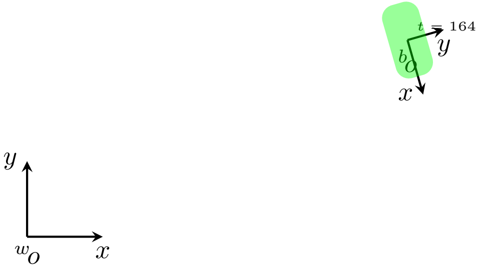

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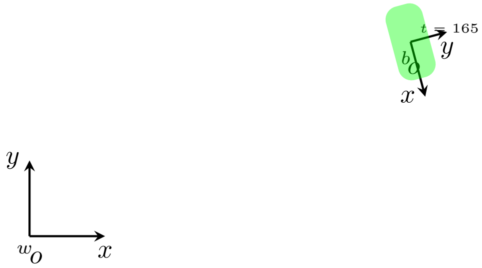

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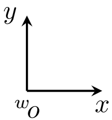
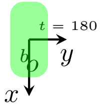
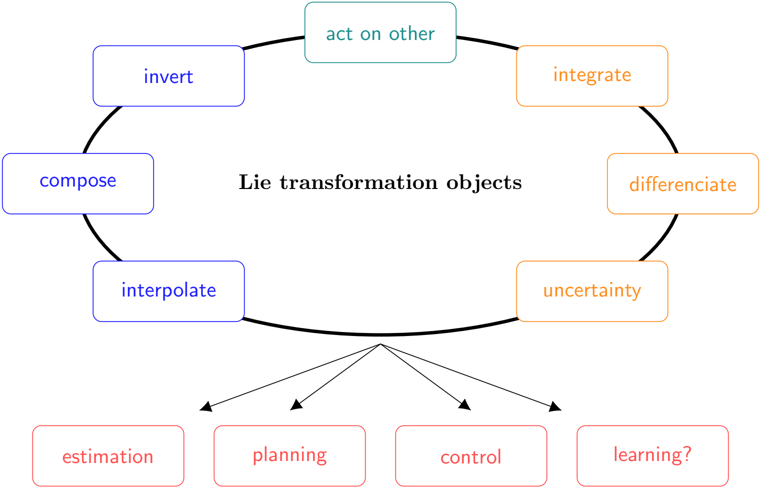




Why Lie groups?

Abstract and principled way to do all this:



Outline

Group

Definition through the four group axioms

- Group: set $\mathcal{G}$ of elements $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots\}$ with an operation '$\circ$' such that:
  - Composition stays in the group: $\mathcal{X} \circ \mathcal{Y} \in \mathcal{G}$
  - Identity element is in the group: $\mathcal{X} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{X} = \mathcal{X}$
  - Inverse element is in the group: $\mathcal{X}^{-1} \circ \mathcal{X} = \mathcal{X} \circ \mathcal{X}^{-1} = \mathcal{E}$
  - Operation is associative: $(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z})$
- In many groups of interest, the operation '$\circ$' is non-commutative!

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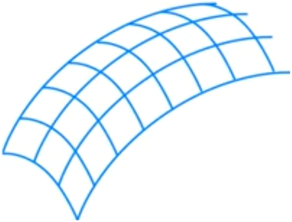
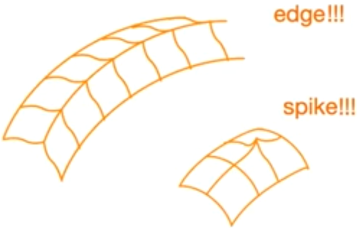
The Lie group

Definition: A group that is also a smooth manifold

Smooth manifold



Non-smooth manifold



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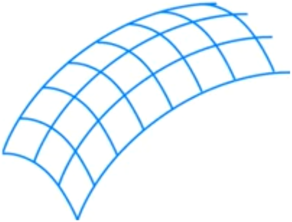

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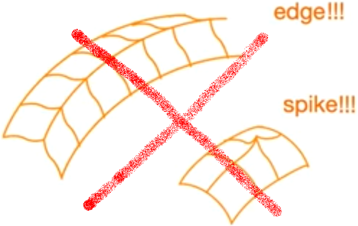

The other definition: A Lie group is a smooth manifold whose elements satisfy the group axioms.

Group action

Definition

- A group can act on another set V to transform its elements
- Given $\mathcal{X}, \mathcal{Y}$ in $\mathcal{G}$ and v in V, the action '$\cdot$' is such that:
  - Identity is the null action: $\mathcal{E} \cdot v = v$
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- Lie groups were formely known as "continuous transformation groups".

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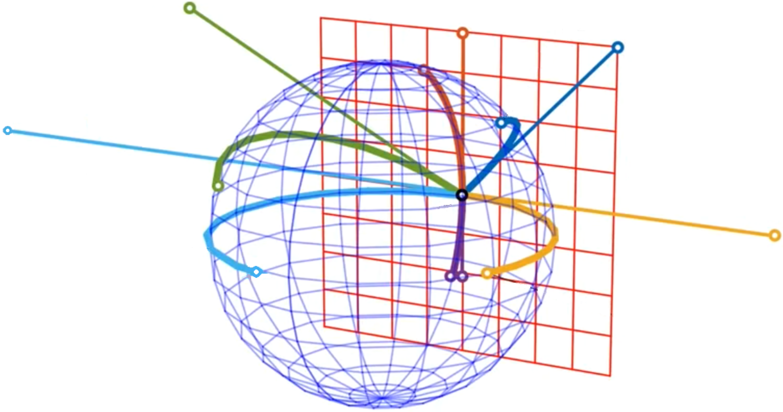
The topology of Lie theory



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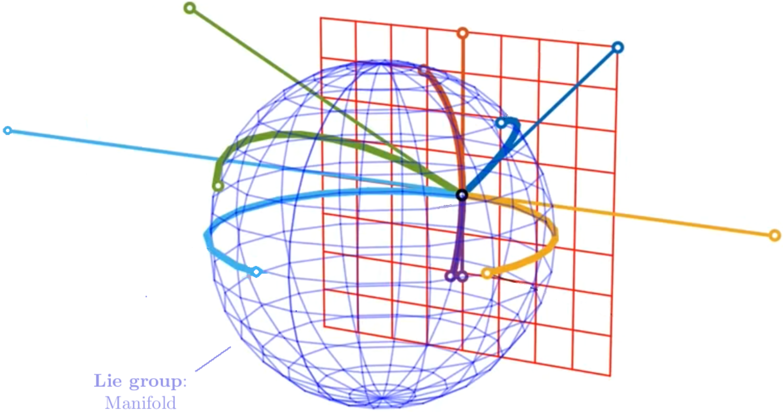

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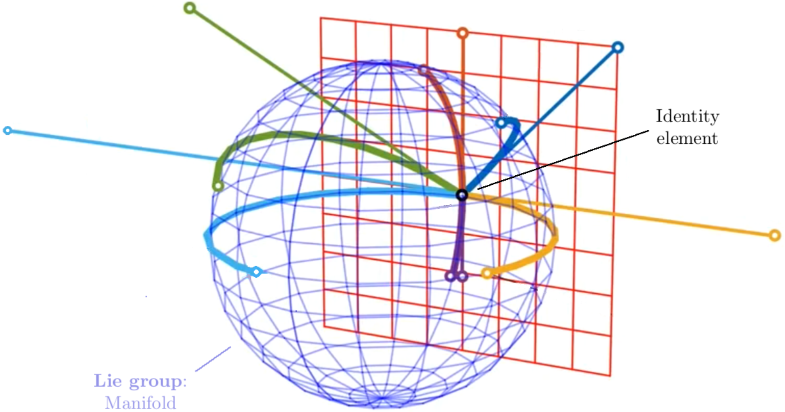

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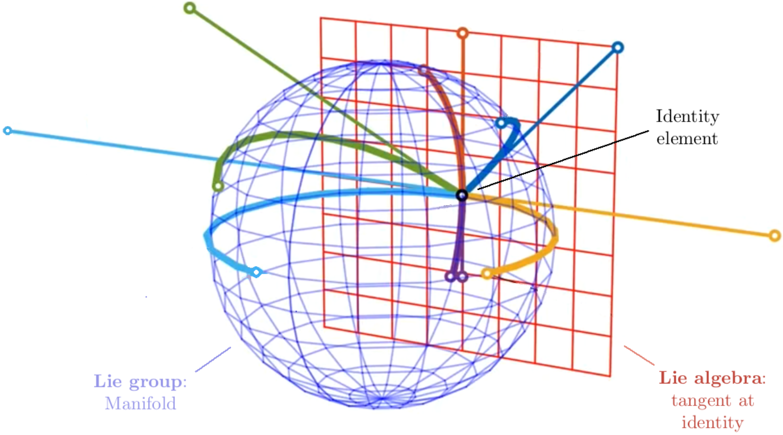

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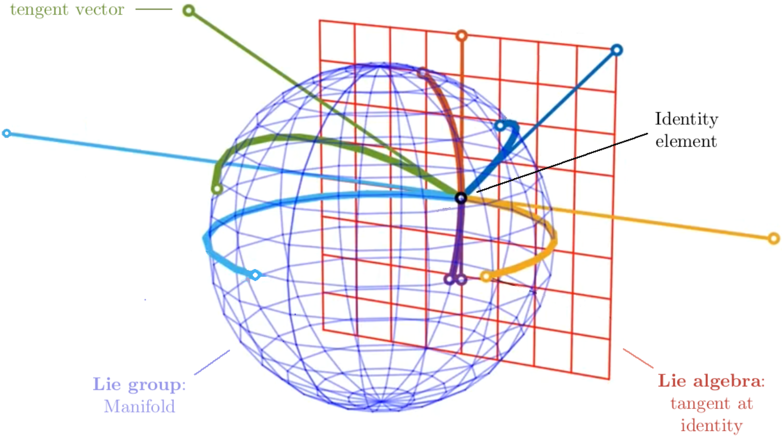

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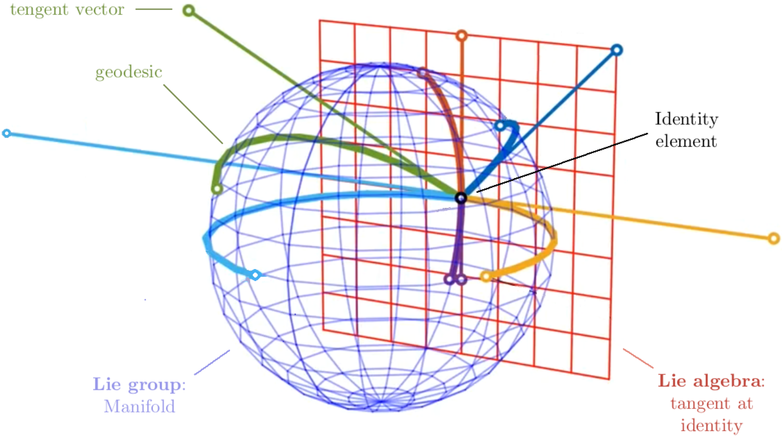

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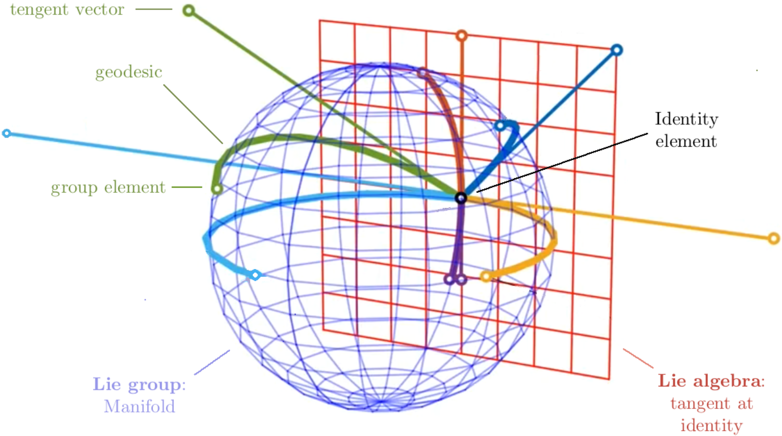

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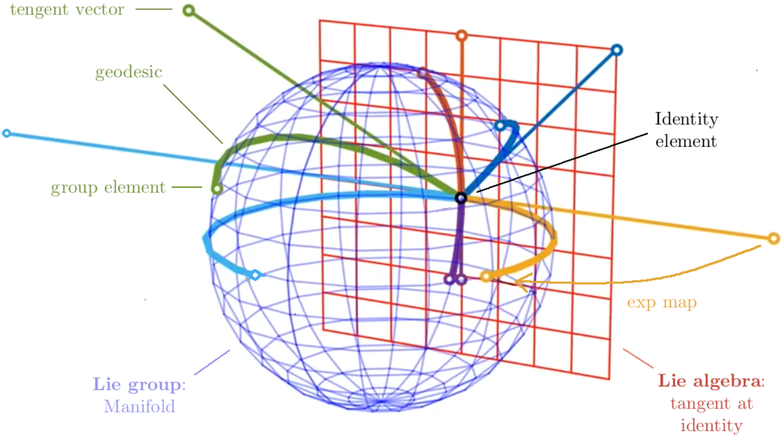

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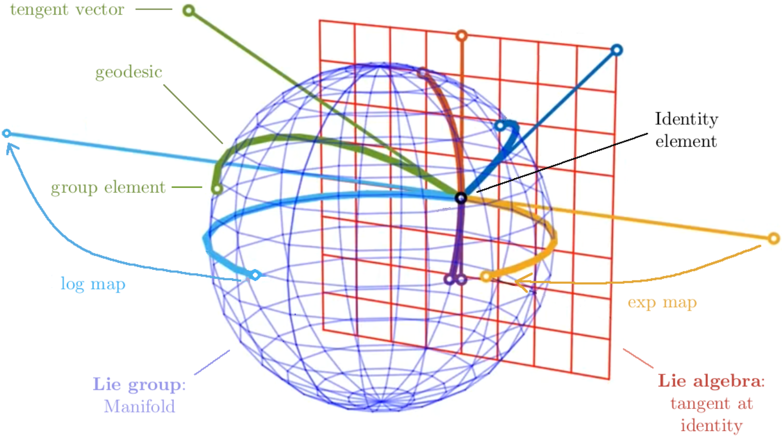
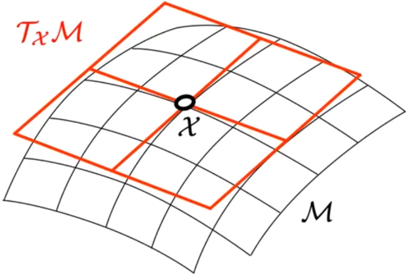


The tangent space and the Lie algebra



- The tangent space at each point is unique
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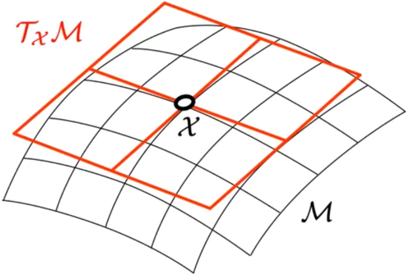

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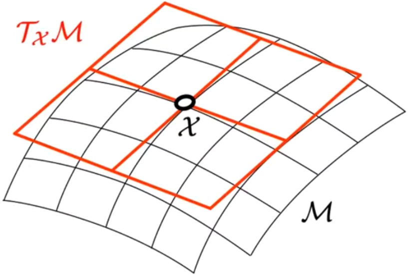

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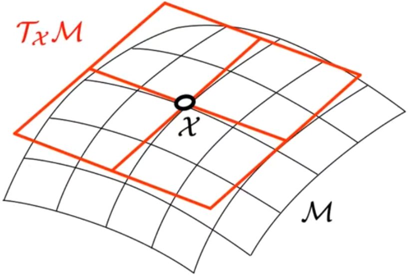

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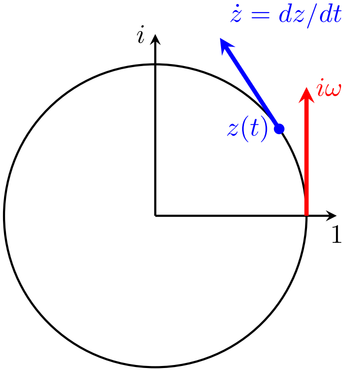
The tangent space of S^1

Structure of the tangent space:
Consider the velocity of a point rotating on the unit circle.



- Differentiate $z^* \cdot z = 1$ w.r.t. time:

$$\dot{z}^* z + z^* \dot{z} = 0$$
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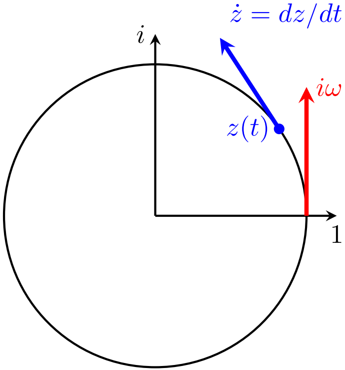

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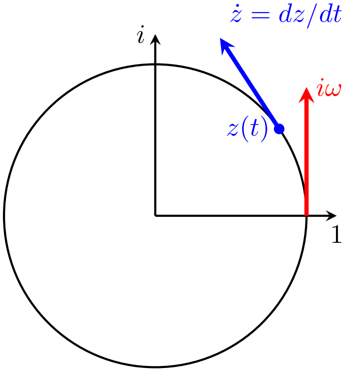

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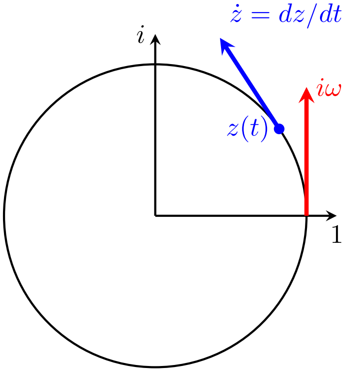

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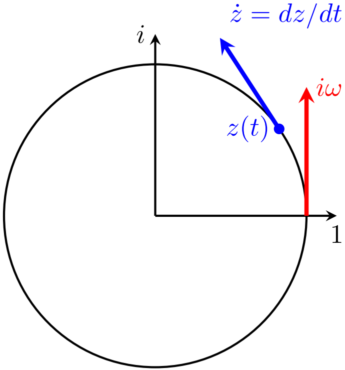

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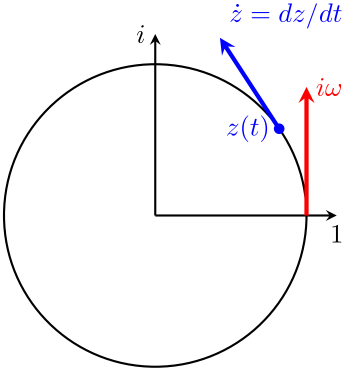

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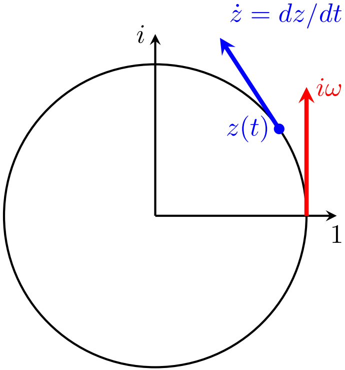

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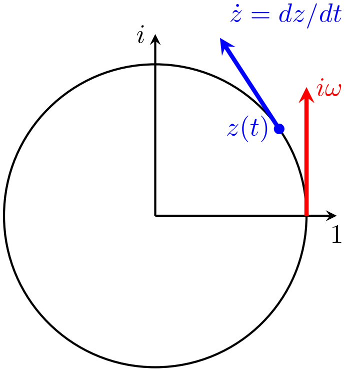

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The tangent space of SO(3)

Structure of the tangent space:
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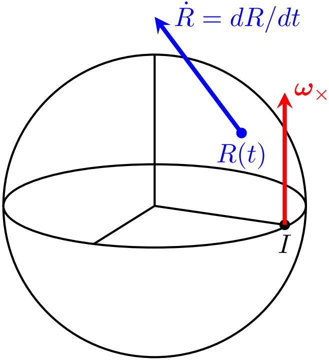

- Differentiate $R^T \cdot R = I$ w.r.t. time:

$$\dot{R}^T R + R^T \dot{R} = 0$$
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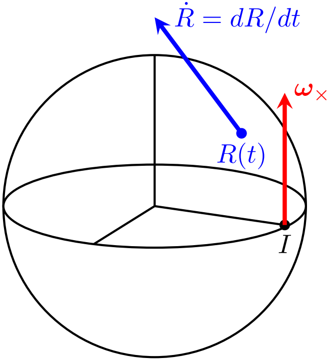

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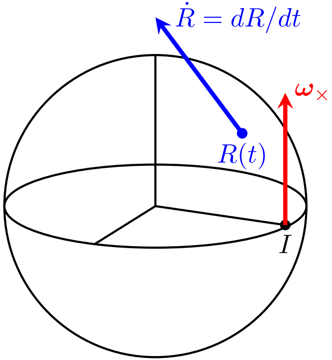

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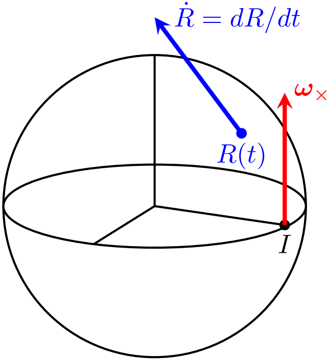

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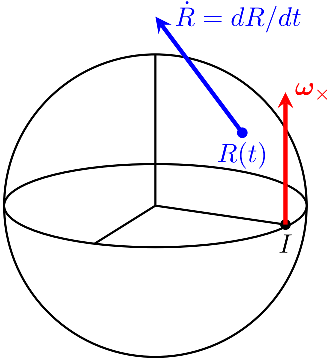

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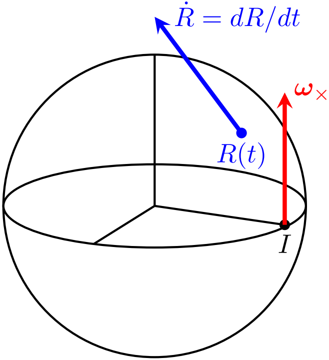

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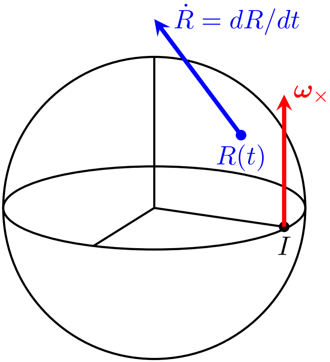
Lie algebra v.s Cartesian representation



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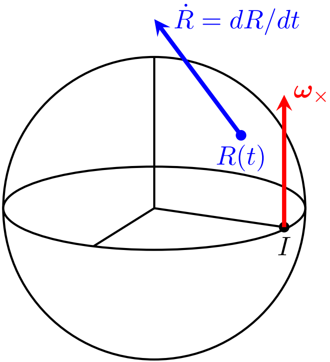

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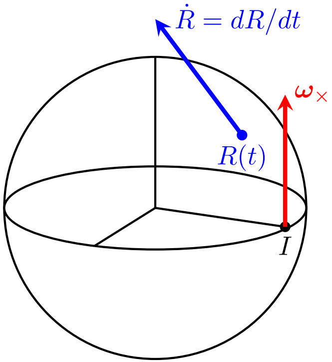

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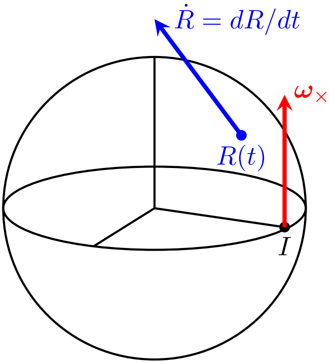

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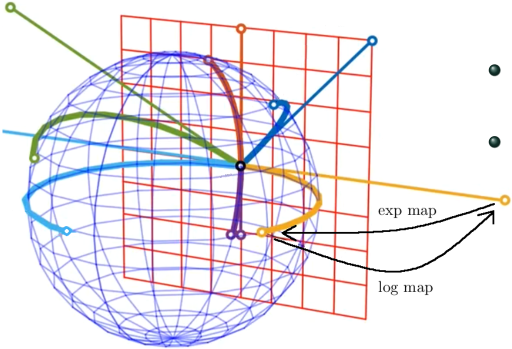
Outline

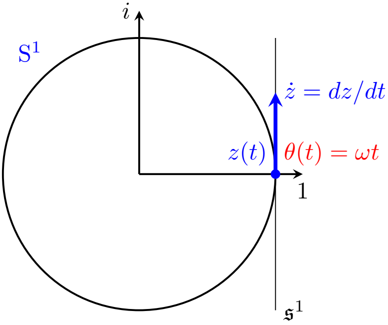
The exponential and logarithmic map



- $\exp$: From tangent to manifold
  - wrap on the geodesic
- $\log$: From manifold to tangent
  - unwrap

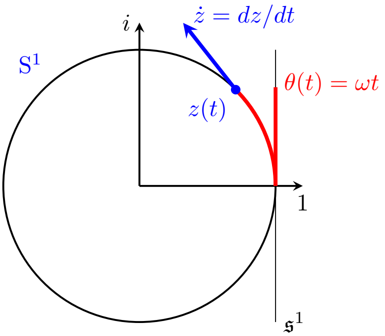
The exponential and logarithmic map

Example: S^1



The exponential and logarithmic map

Example: S^1



- Write ODE and integrate

$$z^* \dot{z} = i\omega \quad \Rightarrow \quad \dot{z} = z i \omega$$
$$z(t) = z_0 \exp(i\omega t)$$
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- If $z_0 = z(0) = 1$ and $i\omega t = i\theta$

$$z(t) = \exp(i\omega t) = \exp(i\theta)$$

- Taylor expansion of $\exp(i\theta)$:

$$\begin{aligned} \exp(i\theta) &= 1 + i\theta + (i\theta)^2/2 + (i\theta)^3/3! + \cdots \\ &= 1 + i\theta - \theta^2/2 - i\theta^3/3! + \theta^4/4! \cdots \\ &= (1 - \theta^2/2 + \cdots) + i(\theta - \theta^3/3! + \cdots) \\ &= \cos\theta + i \sin\theta \end{aligned}$$

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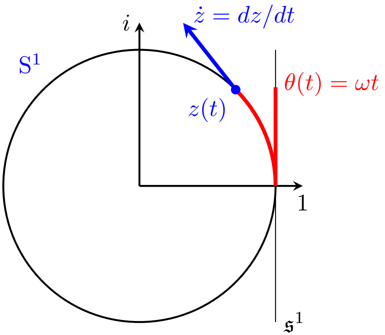

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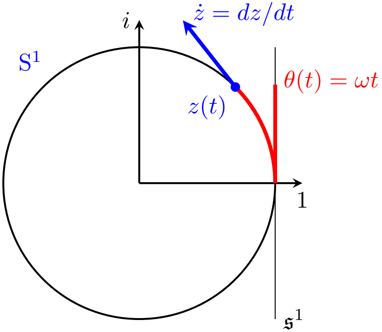

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The exponential and logarithmic map

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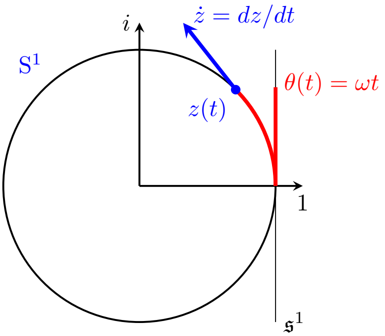

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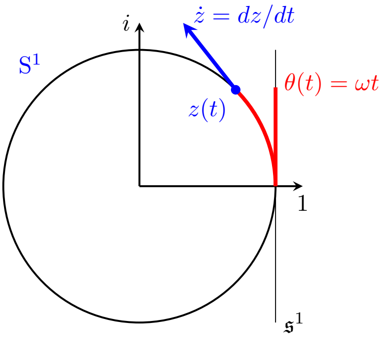

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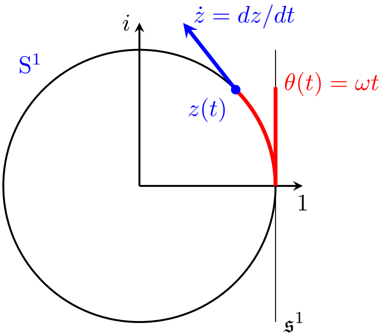

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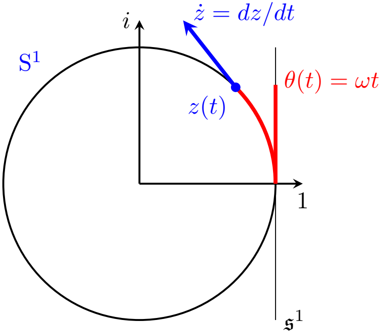

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The exponential and logarithmic map

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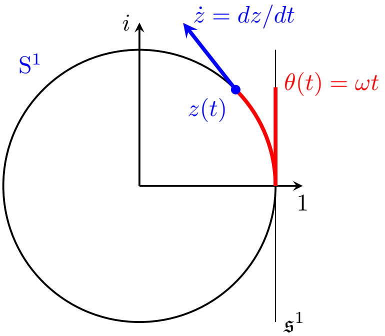

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The exponential and logarithmic map

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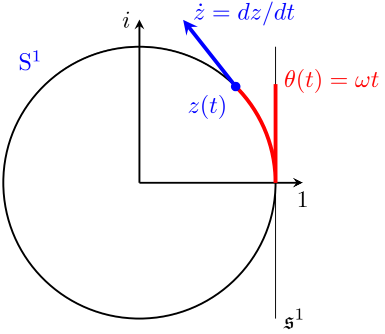

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The exponential and logarithmic map

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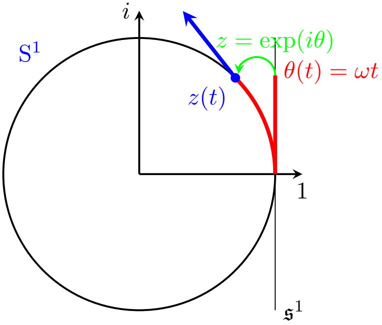

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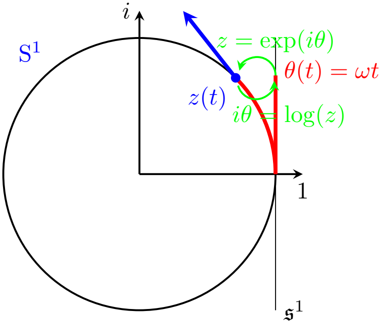

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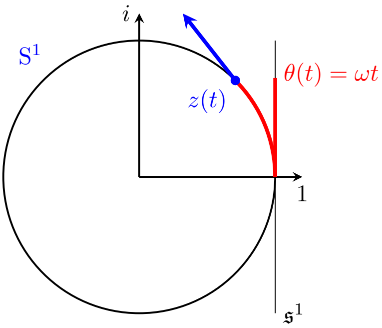

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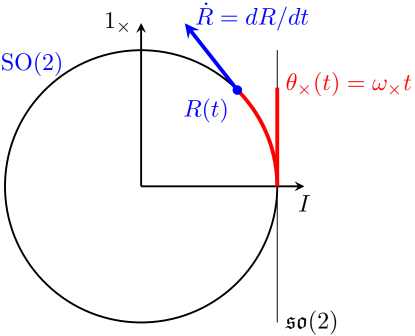
The exponential and logarithmic map

Example: SO(2)



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The exponential and logarithmic map

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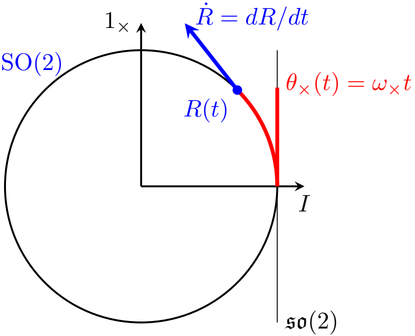

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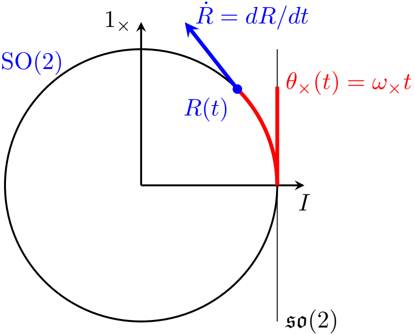

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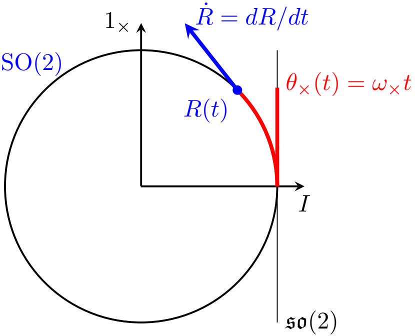

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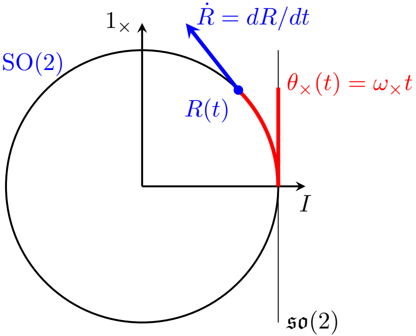

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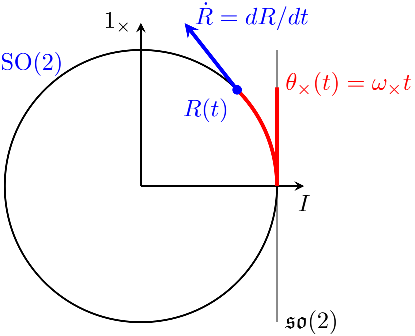

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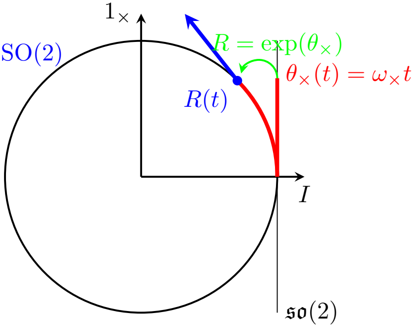

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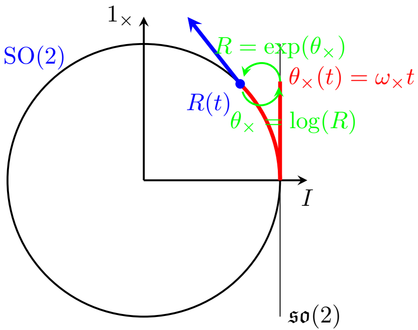

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$$R(t) = \exp(\omega_\times t) = \exp(\theta_\times)$$

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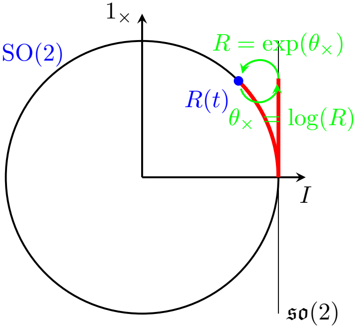
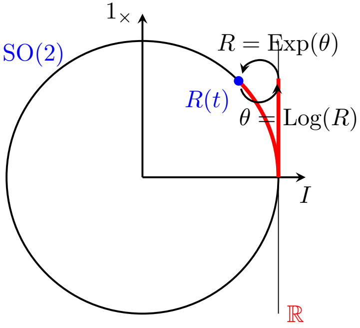
$$\begin{aligned}\exp(\theta_\times) &= I + \theta_\times + (\theta_\times)^2/2 + (\theta_\times)^3/3! + \cdots \\ &= I(1 - \theta^2/2 + \cdots) + 1_\times(\theta - \theta^3/3! + \cdots) \\ &= I\cos\theta + 1_\times \sin\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\end{aligned}$$

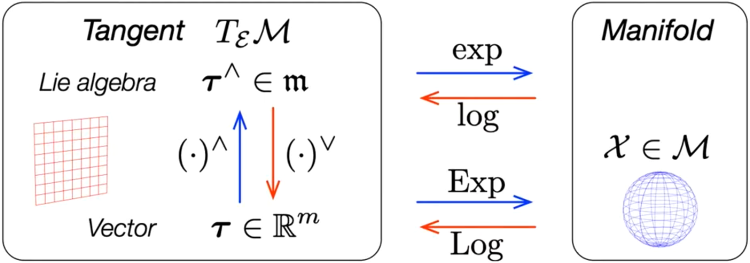
The capitalized exponential and logarithmic map





The capitalized exponential and logarithmic map

Skip the Lie algebra, and work always in Cartesian

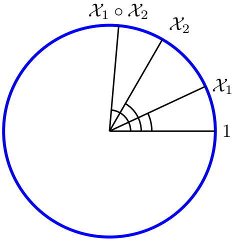


Exp and Log are mere shortcuts, but very useful

Plus and minus operators

The plus operator: right-$\oplus$: $\mathcal{Y} = \mathcal{X} \oplus \omega$ (and left-$\oplus$: $\mathcal{Y} = \omega \oplus \mathcal{X}$)

Composition



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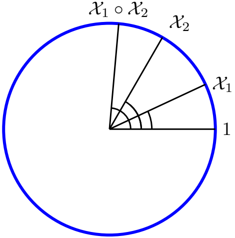
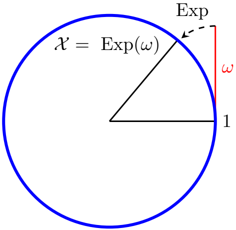
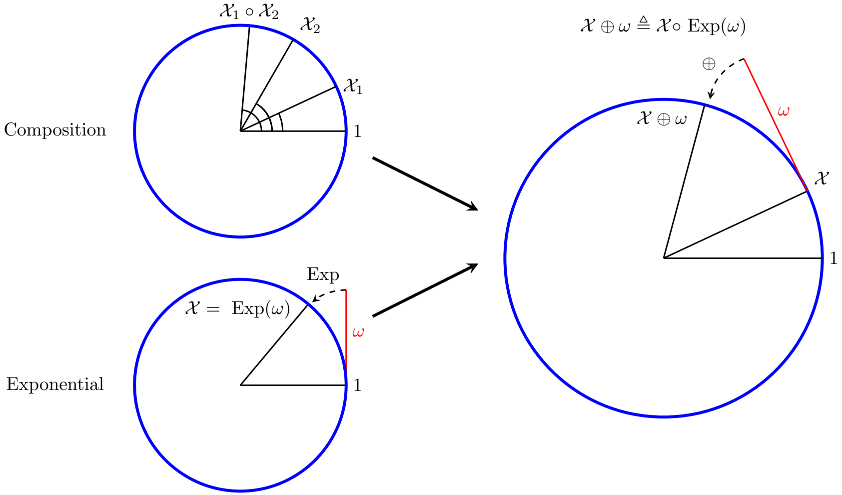


Exponential



Plus and minus operators

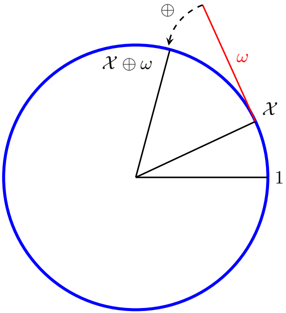
The plus operator: right-$\oplus$: $\mathcal{Y} = \mathcal{X} \oplus \omega$ (and left-$\oplus$: $\mathcal{Y} = \omega \oplus \mathcal{X}$)



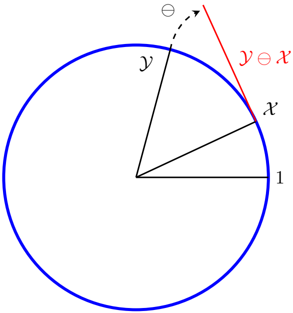
Plus and minus operators

The minus operator: right-$\ominus$: $\mathcal{Y} = \mathcal{X} \ominus \omega$ (and left-$\ominus$: $\mathcal{Y} = \omega \ominus \mathcal{X}$)

$\mathcal{X} \oplus \omega \triangleq \mathcal{X} \circ \operatorname{Exp}(\omega)$

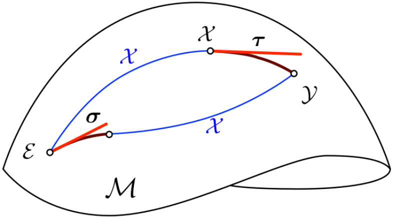


$\mathcal{Y} \ominus \mathcal{X} \triangleq \operatorname{Log}(\mathcal{X}^{-1} \circ \mathcal{Y})$



Plus and minus are alos shortcuts, but also very useful

The adjoint matrix



Note: $\boldsymbol{\sigma} \in T_{\mathcal{E}}\mathcal{M}$ and $\boldsymbol{\tau} \in T_{\mathcal{X}}\mathcal{M}$

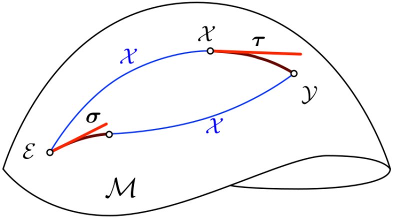
$$\mathcal{Y} = \boldsymbol{\sigma} \oplus \mathcal{X} = \mathcal{X} \oplus \boldsymbol{\tau}$$

$$\Rightarrow \boldsymbol{\sigma}^\wedge = \mathcal{X} \cdot \boldsymbol{\tau}^\wedge \cdot \mathcal{X}^{-1}$$

$$\Rightarrow \boldsymbol{\sigma} = \mathbf{Ad}_{\mathcal{X}} \cdot \boldsymbol{\tau}$$

- Linear: matrix operator
- Maps: $T_{\mathcal{X}}\mathcal{M}$ to $T_{\mathcal{E}}\mathcal{M}$

The adjoint matrix



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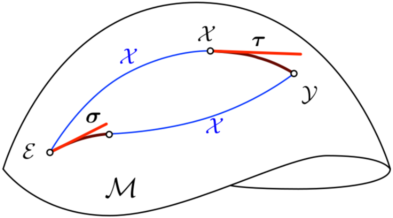
- Maps: $T_{\mathcal{X}}\mathcal{M}$ to $T_{\mathcal{E}}\mathcal{M}$

The adjoint matrix



Note: $\boldsymbol{\sigma} \in T_{\mathcal{E}}\mathcal{M}$ and $\boldsymbol{\tau} \in T_{\mathcal{X}}\mathcal{M}$

$$
\begin{aligned}
\mathcal{Y} = \boldsymbol{\sigma} \oplus \mathcal{X} &= \mathcal{X} \oplus \boldsymbol{\tau} \\
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\end{aligned}
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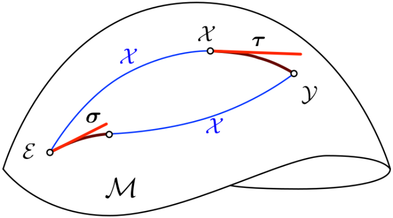
The adjoint matrix



Note: $\boldsymbol{\sigma} \in T_{\mathcal{E}}\mathcal{M}$ and $\boldsymbol{\tau} \in T_{\mathcal{X}}\mathcal{M}$

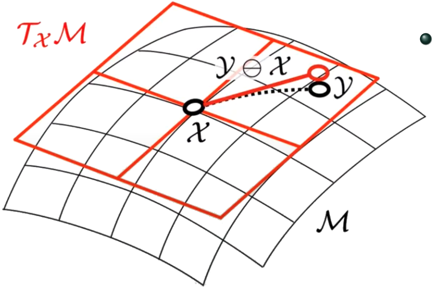
$$\begin{aligned} \mathcal{Y} &= \boldsymbol{\sigma} \oplus \mathcal{X} = \mathcal{X} \oplus \boldsymbol{\tau} \\ &\Rightarrow \boldsymbol{\sigma}^\wedge = \mathcal{X} \cdot \boldsymbol{\tau}^\wedge \cdot \mathcal{X}^{-1} \\ &\Rightarrow \boldsymbol{\sigma} = \mathbf{Ad}_{\mathcal{X}} \cdot \boldsymbol{\tau} \end{aligned}$$

- Linear: matrix operator
- Maps: $T_{\mathcal{X}}\mathcal{M}$ to $T_{\mathcal{E}}\mathcal{M}$

Outline

Calculus on Lie groups

Use the plus and minus operators!



- Express as Cartesian vector:
  - Perturbations, errors, increments, ...
- And define easily:
  - Jacobians of functions $f : \mathcal{M} \to \mathcal{N}$
  - Covariances of elements $\mathcal{X}$ in $\mathcal{M}$

Jacobians on Lie groups

Use the plus and minus operators!

Vector spaces

$$\boldsymbol{J} = \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}}$$
$$= \lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})}{\boldsymbol{h}} \in \mathbb{R}^{n\times m}$$

Lie groups

$$\boldsymbol{J}_r = \frac{D\boldsymbol{f}(\boldsymbol{\mathcal{X}})}{D\boldsymbol{\mathcal{X}}}$$
$$= \lim_{\boldsymbol{\tau}\to\boldsymbol{0}} \frac{f(\mathcal{X}\oplus\boldsymbol{\tau}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \in \mathbb{R}^{n\times m}$$

how to:

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})}{\boldsymbol{h}} = \cdots$$
$$= \lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{\boldsymbol{J}\boldsymbol{h}}{\boldsymbol{h}}$$
$$\triangleq \frac{\partial \boldsymbol{J}\boldsymbol{h}}{\partial \boldsymbol{h}} = \boldsymbol{J}$$

same thing!!!

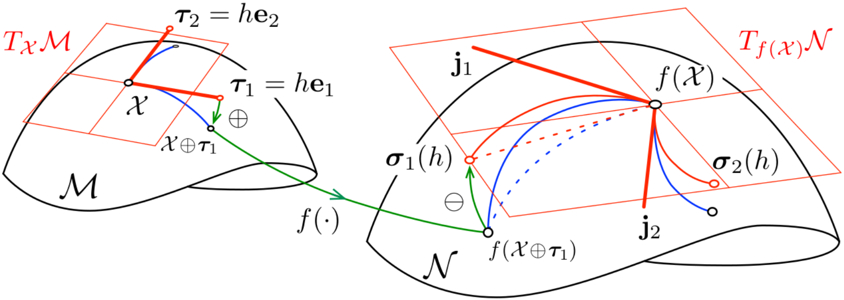
$$\lim_{\boldsymbol{\tau}\to\boldsymbol{0}} \frac{f(\mathcal{X}\oplus\boldsymbol{\tau}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}}$$
$$= \lim_{\boldsymbol{\tau}\to\boldsymbol{0}} \frac{\mathsf{Log}[f^{-1}(\mathcal{X})f(\mathcal{X}\circ\mathsf{Exp}(\boldsymbol{\tau}))]}{\boldsymbol{\tau}}$$
$$= \lim_{\boldsymbol{\tau}\to\boldsymbol{0}} \frac{\boldsymbol{J}_r\boldsymbol{\tau}}{\boldsymbol{\tau}} \triangleq \frac{\partial \boldsymbol{J}_r\boldsymbol{\tau}}{\partial \boldsymbol{\tau}} = \boldsymbol{J}_r$$

Jacobians on Lie groups

Jacobian maps $T_{\mathcal{X}}\mathcal{M}$ to $T_{f(\mathcal{X})}\mathcal{N}$

$$f : \mathcal{M} \rightarrow \mathcal{N}; \mathcal{X} \mapsto \mathcal{Y} = f(\mathcal{X})$$

$$\boldsymbol{J}_r = \frac{Df(\mathcal{X})}{D\mathcal{X}} = \lim_{\boldsymbol{\tau}\rightarrow\mathbf{0}} \frac{f(\mathcal{X} \oplus \boldsymbol{\tau}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \in \mathbb{R}^{n\times m}$$



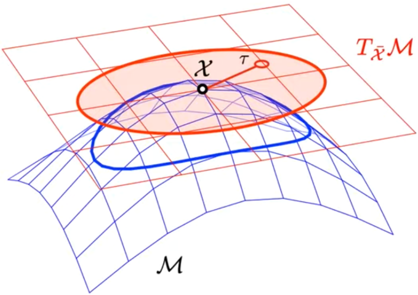
Differentiation rules on Lie groups

From elementary Jacobian blocks to any Jacobian

For each group		For all groups		Deduce from the previous	
adjoint	$\mathbf{Ad}_{\mathcal{X}}$	inverse	$\dfrac{D\mathcal{X}^{-1}}{D\mathcal{X}} = -\mathbf{Ad}_{\mathcal{X}}$	Log	$\dfrac{D\mathrm{Log}(\mathcal{X})}{D\mathcal{X}} = \boldsymbol{J}_r^{-1}(\mathrm{Log}(\mathcal{X}))$
right Jacobian	$\boldsymbol{J}_r = \dfrac{D\mathrm{Exp}(\boldsymbol{\tau})}{D\boldsymbol{\tau}}$	comp.	$\dfrac{D\mathcal{X}\circ\mathcal{Y}}{D\mathcal{X}} = \mathbf{Ad}_{\mathcal{Y}}^{-1}$	plus	$\dfrac{D\mathcal{X}\oplus\boldsymbol{\tau}}{D\mathcal{X}} = \mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})}^{-1}$
action	$\dfrac{D\mathcal{X}\cdot\boldsymbol{p}}{D\mathcal{X}}, \dfrac{D\mathcal{X}\cdot\boldsymbol{p}}{D\boldsymbol{p}}$		$\dfrac{D\mathcal{X}\circ\mathcal{Y}}{D\mathcal{Y}} = I$		$\dfrac{D\mathcal{X}\oplus\boldsymbol{\tau}}{D\boldsymbol{\tau}} = \boldsymbol{J}_r(\boldsymbol{\tau})$

Use the chain rule for any other Jacobian!

Perturbations on Lie groups and covariance matrices



- Perturbation $\boldsymbol{\tau}$ over $\mathcal{X}$:

$$\mathcal{X} = \bar{\mathcal{X}} \oplus \boldsymbol{\tau}$$

- Covariance of $\mathcal{X}$:

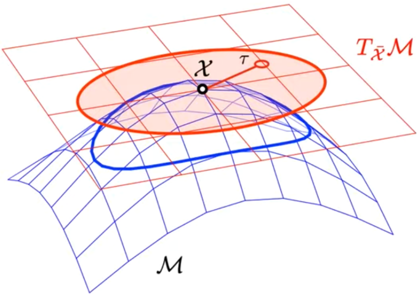
$$\begin{aligned} \Sigma &\stackrel{\text{def}}{=} \mathrm{E}(\boldsymbol{\tau} \cdot \boldsymbol{\tau}^T) \\ &= \mathrm{E}[(\mathcal{X} \ominus \bar{\mathcal{X}}) \cdot (\mathcal{X} \ominus \bar{\mathcal{X}})^T] \end{aligned}$$

- Propagation is easy!

$$\mathcal{Y} = f(\mathcal{X}) \qquad \boldsymbol{J} = \frac{D\mathcal{Y}}{D\mathcal{X}}$$

$$\Rightarrow \Sigma_{\mathcal{Y}} = \boldsymbol{J} \cdot \Sigma_{\mathcal{X}} \cdot \boldsymbol{J}^T$$

Perturbations on Lie groups and covariance matrices



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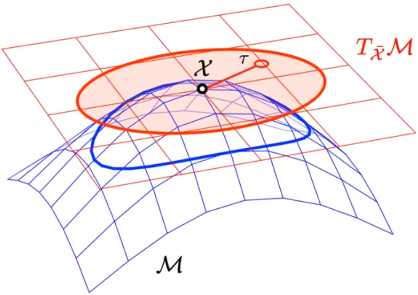
$$\Rightarrow \boldsymbol{\Sigma}_{\mathcal{Y}} = \boldsymbol{J} \cdot \boldsymbol{\Sigma}_{\mathcal{X}} \cdot \boldsymbol{J}^T$$

Perturbations on Lie groups and covariance matrices



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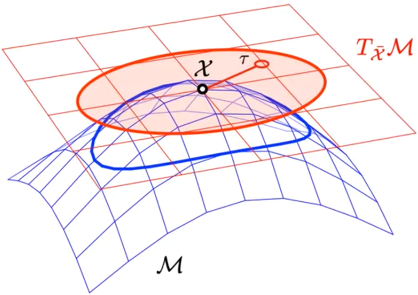
Perturbations on Lie groups and covariance matrices



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$$\mathcal{Y} = f(\mathcal{X}) \qquad \boldsymbol{J} = \frac{D\mathcal{Y}}{D\mathcal{X}}$$

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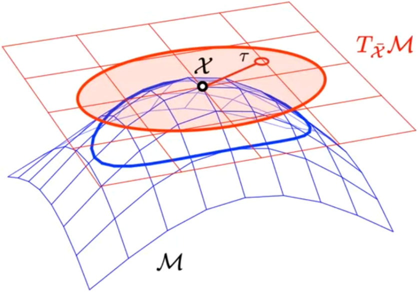
Perturbations on Lie groups and covariance matrices



- Perturbation $\boldsymbol{\tau}$ over $\mathcal{X}$:

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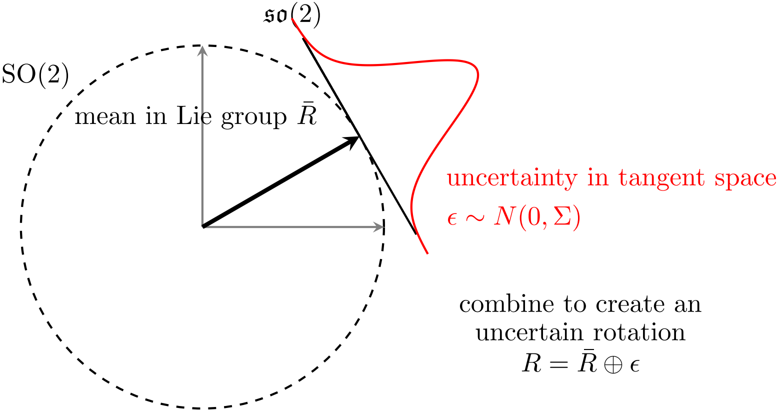
- Propagation is easy!

$$\mathcal{Y} = f(\mathcal{X}) \qquad \boldsymbol{J} = \frac{D\mathcal{Y}}{D\mathcal{X}}$$

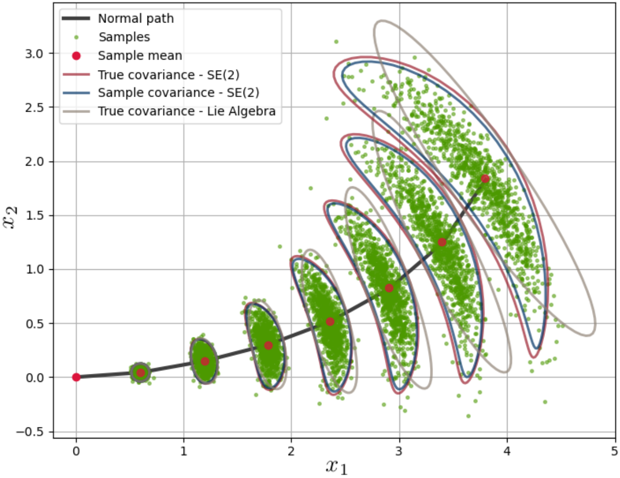
$$\Rightarrow \boldsymbol{\Sigma}_{\mathcal{Y}} = \boldsymbol{J} \cdot \boldsymbol{\Sigma}_{\mathcal{X}} \cdot \boldsymbol{J}^T$$

Gaussian random variables and PDFs



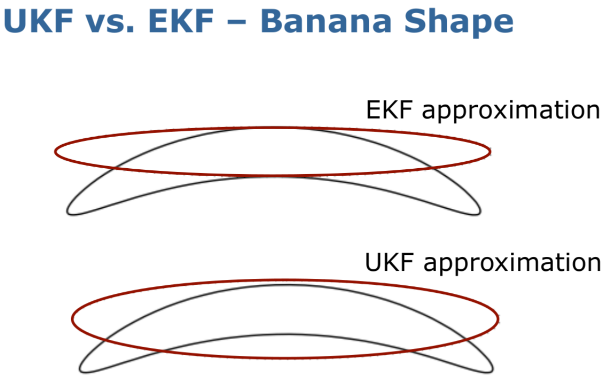
Banana shape is Gaussian on the tangent space: SE(2)

Noisy process: $\mathcal{X}_{t+1} = \mathcal{X}_t \cdot \mathrm{Exp}(\boldsymbol{u}_t) \oplus \boldsymbol{\epsilon}_t$,
where $\mathcal{X}_t \in \mathrm{SE}(2)$, $\boldsymbol{u}_t, \boldsymbol{\epsilon}_t \in \mathbb{R}^3$ ($\boldsymbol{\epsilon}_t^\wedge \in \mathfrak{se}(2)$), and $\boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma})$



Compare with traditional EKF and UKF approach

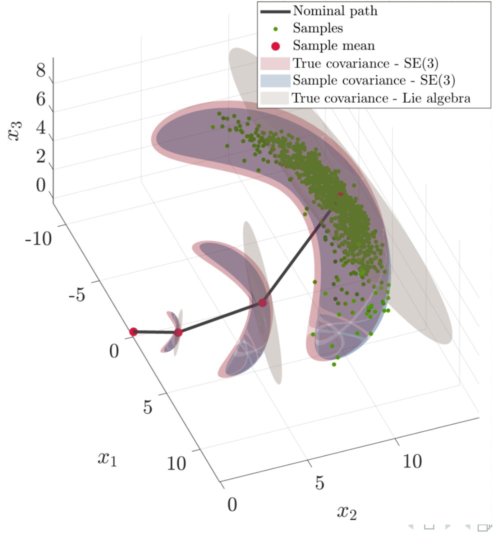
Noisy process: $\boldsymbol{x}_{t+1} = f(\boldsymbol{x}_t, \boldsymbol{u}_t) + \boldsymbol{\epsilon}_t$, where f is a non-linear function and $\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{\epsilon}_t \in \mathbb{R}^3$



Courtesy by Stachniss, C. Introduction to Robot Mapping. Winter Semeter, 2012.

Banana shape is Gaussian on the tangent space: SE(3)

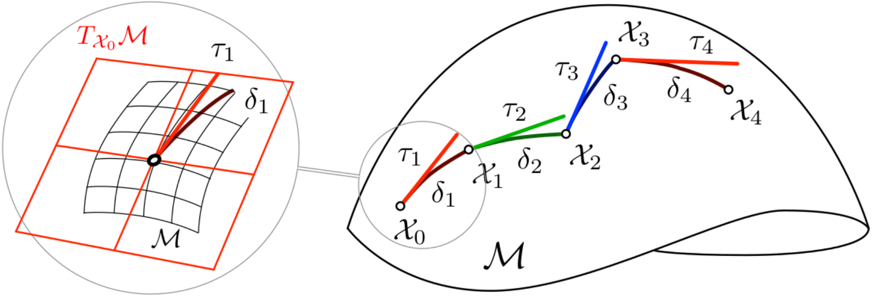
Noisy process: $\mathcal{X}_{t+1} = \mathcal{X}_t \cdot \mathrm{Exp}(\boldsymbol{u}_t) \oplus \boldsymbol{\epsilon}_t$,
where $\mathcal{X}_t \in \mathrm{SE}(3)$, $\boldsymbol{u}_t, \boldsymbol{\epsilon}_t \in \mathbb{R}^6$ $(\boldsymbol{\epsilon}_t^\wedge \in \mathfrak{se}(3))$, and $\boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma})$



Integration on Lie groups

continuous time, $\boldsymbol{\omega}$ const

discrete time, $\boldsymbol{\omega}$ piecewise constant



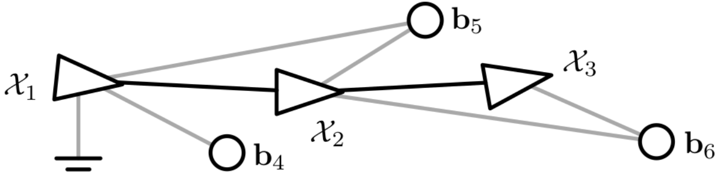
$\mathcal{X}(t) = \mathcal{X}_0 \cdot \mathrm{Exp}(\boldsymbol{\omega} t)$

$\mathcal{X}_4 = \mathcal{X}_0 \oplus (\boldsymbol{\omega}_1 dt) \oplus (\boldsymbol{\omega}_2 dt) \oplus (\boldsymbol{\omega}_3 dt) \oplus (\boldsymbol{\omega}_4 dt)$

Note: $\boldsymbol{\tau} = \boldsymbol{\omega} dt$ and $\boldsymbol{\delta} = \mathrm{Exp}(\boldsymbol{\tau})$

Outline

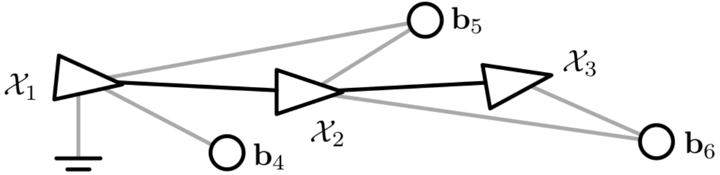
Application for state estimation



Courtesy by Solà, J., Dery, J., and Atchuthan, D. (2021). A micro Lie theroy for state estimation in robotics.

- Poses (unknown): $\mathcal{X} \sim N(\bar{\mathcal{X}}, \mathbf{\Sigma}) \in$ SE(2) (or SE(3))
- Landmarks: $\boldsymbol{b}_k \in \mathbb{R}^2$ (or $\mathbb{R}^3$)
  - if landmarks are known → KF-Based Localization
  - if landmarks are unknown → Graph-Based SLAM (Skip! Next time!)

Application for state estimation



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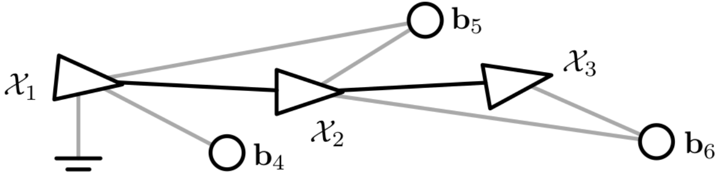
Application for state estimation



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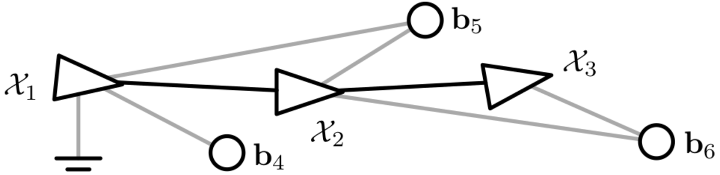

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Error State KF localization

Motion model:

- Always use right-$\oplus$ for state prediction:

$$\begin{aligned}\mathcal{X}_t &= f(\mathcal{X}_{t-1}, \boldsymbol{u}_t, \boldsymbol{\epsilon}_t) \\ &= \mathcal{X}_{t-1} \oplus (\boldsymbol{u}_t + \boldsymbol{\epsilon}_t)\end{aligned}$$

  where $\boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \boldsymbol{Q}_t)$ is the perturbation.

- Taylor expansion at $\mathcal{X}_{t-1} = \bar{\mathcal{X}}_{t-1}$ and $\boldsymbol{\epsilon}_t = \mathbf{0}$, we have

$$\mathcal{X}_t = \bar{\mathcal{X}}_t \oplus \boldsymbol{F}_t(\mathcal{X}_{t-1} \ominus \bar{\mathcal{X}}_{t-1}) \oplus \boldsymbol{W}_t \boldsymbol{\epsilon}_t,$$

  where $\boldsymbol{F}_t = \frac{Df}{D\mathcal{X}_t}$ and $\boldsymbol{W}_t = \frac{Df}{D\boldsymbol{\epsilon}_t}$ are jacobians.

- Define the error $\boldsymbol{\xi}_t = \mathcal{X}_t \ominus \bar{\mathcal{X}}_t$, then

$$\boldsymbol{\xi}_t = \boldsymbol{F}_t \boldsymbol{\xi}_{t-1} + \boldsymbol{W}_t \boldsymbol{\epsilon}_t$$

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- Always use right-$\oplus$ for state prediction:

$$\begin{aligned}\mathcal{X}_t &= f(\mathcal{X}_{t-1}, \boldsymbol{u}_t, \boldsymbol{\epsilon}_t) \\ &= \mathcal{X}_{t-1} \oplus (\boldsymbol{u}_t + \boldsymbol{\epsilon}_t)\end{aligned}$$

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- Taylor expansion at $\mathcal{X}_{t-1} = \bar{\mathcal{X}}_{t-1}$ and $\boldsymbol{\epsilon}_t = \mathbf{0}$, we have

$$\mathcal{X}_t = \bar{\mathcal{X}}_t \oplus \boldsymbol{F}_t(\mathcal{X}_{t-1} \ominus \bar{\mathcal{X}}_{t-1}) \oplus \boldsymbol{W}_t \boldsymbol{\epsilon}_t,$$

  where $\boldsymbol{F}_t = \frac{Df}{D\mathcal{X}_t}$ and $\boldsymbol{W}_t = \frac{Df}{D\boldsymbol{\epsilon}_t}$ are jacobians.

- Define the error $\boldsymbol{\xi}_t = \mathcal{X}_t \ominus \bar{\mathcal{X}}_t$, then

$$\boldsymbol{\xi}_t = \boldsymbol{F}_t \boldsymbol{\xi}_{t-1} + \boldsymbol{W}_t \boldsymbol{\epsilon}_t$$

Error State KF localization

Motion model:

- Always use right-$\oplus$ for state prediction:

$$\begin{aligned}\mathcal{X}_t &= f(\mathcal{X}_{t-1}, \boldsymbol{u}_t, \boldsymbol{\epsilon}_t) \\ &= \mathcal{X}_{t-1} \oplus (\boldsymbol{u}_t + \boldsymbol{\epsilon}_t)\end{aligned}$$

  where $\boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \boldsymbol{Q}_t)$ is the perturbation.

- Taylor expansion at $\mathcal{X}_{t-1} = \bar{\mathcal{X}}_{t-1}$ and $\boldsymbol{\epsilon}_t = \mathbf{0}$, we have

$$\mathcal{X}_t = \bar{\mathcal{X}}_t \oplus \boldsymbol{F}_t(\mathcal{X}_{t-1} \ominus \bar{\mathcal{X}}_{t-1}) \oplus \boldsymbol{W}_t \boldsymbol{\epsilon}_t,$$

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Error State KF localization

- Predict the state by the motion model:

$$\check{\boldsymbol{\xi}}_t = \mathbf{0} \qquad \text{predicted error state}$$

$$\check{\mathcal{X}}_t = \bar{\mathcal{X}}_t = \bar{\mathcal{X}}_{t-1} \oplus \boldsymbol{u}_t \qquad \text{predicted nomial state}$$

$$\check{\boldsymbol{\Sigma}}_t = \boldsymbol{F}_t \hat{\boldsymbol{\Sigma}}_{t-1} \boldsymbol{F}_t^T + \boldsymbol{W}_t \boldsymbol{Q}_t \boldsymbol{W}_t^T \qquad \text{predicted error covariance}$$

- The perturbation $\boldsymbol{\epsilon}_t$ has been propagate to the world frame, so the covariance $\check{\boldsymbol{\Sigma}}_t$ is in the world frame.
- We have derived the exact same prediction results as the Invariant Extended Kalman Filter (IEKF)!

Error State KF localization

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Error State KF localization

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Error State KF localization

Measurement model for measurement in the body frame:

- Use the left-$\oplus$ operation if the measurement is taken in the body frame, such as landmark observations from LiDAR or camera sensors.
- Measurements $\boldsymbol{y}_t \in \mathbb{R}^2$ (or $\mathbb{R}^3$) have this from:

$$\boldsymbol{y}_t = \mathcal{X}_t^{-1} \cdot \boldsymbol{b} + \boldsymbol{\delta}_t,$$

  where $\boldsymbol{\delta}_t \sim N(\mathbf{0}, \boldsymbol{R}_t)$.
- Define the innovation $\boldsymbol{z}_t$ such that

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Error State KF localization

- The noise error $\boldsymbol{\delta}_t$ is from the body frame, we will have to switch the covariance from body frame to world frame.
- ${}^{w}\boldsymbol{\xi}_t = \mathbf{Ad}_{\bar{\mathcal{X}}} \cdot {}^{b}\boldsymbol{\xi}_t \;\Rightarrow\; {}^{w}\boldsymbol{\Sigma}_t = \mathbf{Ad}_{\bar{\mathcal{X}}} \cdot {}^{b}\boldsymbol{\Sigma}_t \cdot \mathbf{Ad}_{\bar{\mathcal{X}}}^T$
- Update the state by the measurement model:

$$
\begin{aligned}
\boldsymbol{z}_t &= \bar{\mathcal{X}}_t \cdot (\boldsymbol{y}_t - \bar{\boldsymbol{y}}_t) && \text{innovation} \\
\boldsymbol{S}_t &= \boldsymbol{H}_t \check{\boldsymbol{\Sigma}}_t \boldsymbol{H}_t^T + \boldsymbol{V}_t \boldsymbol{R}_t \boldsymbol{V}_t^T && \text{innovation covariance} \\
\boldsymbol{K}_t &= \check{\boldsymbol{\Sigma}}_t \boldsymbol{H}_t^T \boldsymbol{S}_t^{-1} && \text{Kalman gain} \\
\hat{\boldsymbol{\xi}}_t &= \boldsymbol{K}_t \boldsymbol{z}_t && \text{updated error state} \\
\hat{\mathcal{X}}_t &= \hat{\boldsymbol{\xi}}_t \oplus \bar{\mathcal{X}}_t && \text{updated nominal state} \\
\hat{\boldsymbol{\Sigma}}_t &= (I - \boldsymbol{K}_t \boldsymbol{H}_t) \check{\boldsymbol{\Sigma}}_t && \text{updated error covariance}
\end{aligned}
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- Switching the covariance twice during the update step is very costly. A more efficient approach is to express the covariance in the body frame during the prediction step.

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Error State KF localization

Measurement model for measurement in the world frame:

- Use the right-$\oplus$ operation if the measurement is position measurement in the world frame from a GPS receiver.
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Outline

Conclusion

- We begin with an introduction to Lie theory, emphasizing its foundational role in advanced robotics applications.
- Key theoretical concepts are covered, including Lie groups, manifolds, tangent spaces, Lie algebras, exponential and logarithmic maps, and adjoint matrices.
- Practical applications are introduced, focusing on integration into calculus for operations such as derivatives, Jacobians, and uncertainty modeling, including perturbations, covariance handling, and integration.
- Simulations and real-world examples are demonstrated for SE(2) and SE(3) using ESKF localization techniques.

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Problems

- The motion model is typically expressed in continuous form as:

$$\begin{aligned}\frac{d}{dt}\mathcal{X}_t &= f_{\boldsymbol{u}_t}(\mathcal{X}_t) \\ &= \mathcal{X}_t \boldsymbol{u}_t^{\wedge}\end{aligned}$$

  How do we compute the Jacobians for this continuous ODE?

- If the control input $\boldsymbol{u}_t$ depends on the state $\mathcal{X}_t$, then we need to calculate:

$$\boldsymbol{F}_t = \frac{D\mathcal{X}_t}{D\mathcal{X}_{t-1}} + \frac{D\mathcal{X}_t}{D\boldsymbol{u}_t}\frac{D\boldsymbol{u}_t}{D\mathcal{X}_{t-1}},$$

  where the Jacobian term $\frac{D\boldsymbol{u}_t}{D\mathcal{X}_{t-1}}$ is unclear to me.

- In the $\mathrm{SE}_k(3)$ setting, incorporating sensor bias into the state matrix breaks the group-affine property. What is the appropriate method to address and resolve this issue?

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Thank You!